Riemann Sphere analytics

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Abstract

The Riemann sphere (RS), also know as the *extended plane*, was a breakthrough in complex analysis, introduced in B. Riemann's Doctorial thesis (1851). His presentation was geometrical. We recall the formula for stereographic projection from the Riemann sphere to $\mathbb{C}$, and we derive a formula for its inverse. This is a mapping from Z to $P(x, y, z)$. We then discuss the physical interpretation of the inverse mapping when the complex variable denotes an impedance.[1]

1 Introduction

Here we derive the mapping from a point on the *finite plane* Z to its "image" on the Riemann Sphere **S**. We then inteperprete the meaning of this transformation when the plane defines an impedance $Z(s)$ as a function of the complex frequency variable $s = \sigma + i\omega$.

There are two sets of coordinates required to set up this problem. First there is any point in $\mathbb{R}^3$ denoted $R \equiv [x, y, z]$. The *North Pole* is given by $[0, 0, 1]$ and the *South Pole* as $[0, 0, -1]$. Second the points $Z = X + iY$ on the finite plane ($z = 0$) are $X = x$ and $Y = y$. The points on the *extended plane* are a subset of R, denoted $P(x, y, z)$, such that $||P|| = 1$.

The mapping from the sphere to the finite plane Z, defined as $Z = P^{-1}(x, y, z)$, may be expressed in either rectangular (x, y, z) or in spherical (ϕ, θ) coordinates as[2]

$$Z(x, y, z) = \frac{x + iy}{1 - z} = \cot\left(\frac{\phi}{2}\right) e^{i\theta}. \tag{1}$$

as shown in Fig. 1.[3] We desire the mapping from Z to $[x, y, z]$ on the unit sphere (i.e., $\alpha = P(A)$ of Fig. 1).

The spherical $\cot(\phi/2)$ formula comes from the "law of cotangents" described in Appendix A.

The problem then is to determine $P(Z)$ ($[x, y, z]$ given Z), namely find the mapping from any point Z on the finite Z plane (indicated as A in Fig. 1), to the corresponding "puncture point" coordinates on **S** $\alpha = P$. Formally we may define this mapping as $[x, y, z] = P(Z)$. In other words, given a point Z on the finite plane, determine the points $[x, y, z]$ on **S**, such that $||[x, y, z]|| = 1$.

[1]Eventually we hope to discuss the Mobius transformation of the plane to the sphere.
[2]wikipedia.org/wiki/Riemann_sphere
[3]Jean-Christophe BENOIST wikipedia.org/wiki/Riemann_sphere

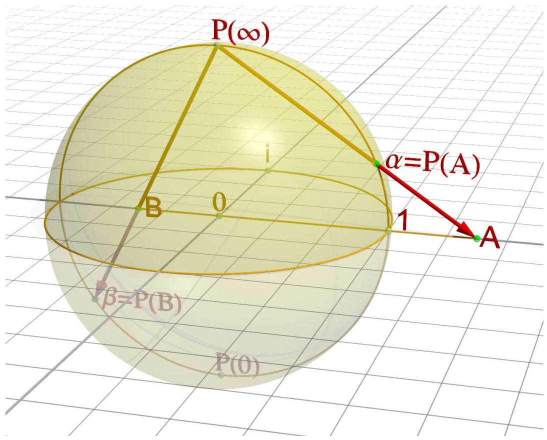

Figure 1: Riemann Sphere

The solution: The final result is[4]

$$[x, y, z] = P(Z) = \frac{[2X, 2Y, |Z|^2 - 1]}{|Z|^2 + 1}, \tag{2}$$

where $X = \Re Z$ and $Y = \Im Z$.

A more compact way of stating $P(Z)$ is to express P in terms of a complex number ζ, proportional to Z

$$\zeta = x + iy = \frac{2Z}{|Z|^2 + 1} \tag{3}$$

along with the corresponding z coordinate

$$z = \frac{|Z|^2 - 1}{|Z|^2 + 1}. \tag{4}$$

Equations 1-4 "make sense" in terms of the construction of Fig. 1:

- Eq. 1 and Eq. 3: $\theta = \angle Z(x, y) = \angle \zeta$. From Eq. 3 we see that $|Z/\zeta| = (1 + |Z|^2)/2$. Thus when $|Z| \geq 1$, $|Z/\zeta| \geq 1$. From the construction this is easy to visualize, as $|\zeta|$ is always inside the unit disk. Less obvious is what happens to $|\zeta|$ for $|Z| < 1$.

- Eq. 2: This equation describes the coordinates for α in terms of Z, whereas Eq. 1 is the inverse relationship.

- Eq. 4 is the "height" of point $\alpha(|Z|)$. When $|Z| = 0$, $z = -1$. When $|Z| = 1$, $z = 0$, and when $|Z| \to \infty$, $z \to 1$

1.1 Mappings between the finite and extended planes

We are looking for the formula for the image point α given any point $Z = X + iY$ on the finite plane. The approach is to derive the formula for the mapping from the north pole of **S** to any point $R \in \mathbb{R}^2$.

[4] http://www.encyclopediaofmath.org/index.php/Riemann_sphere

A line $R(t) = p + t(q - p)$ is defined by two points $p, q \in \mathbb{R}^3$. When $t = 0$, $R(0) = p$ and when $t = 1$, $R(1) = q$. The line from the north pole $p = [0, 0, 1]$ to point $q = [x, y, z]$ (any point in $\mathbb{R}^3$) is thus given by

$$R(t) = [tx, ty, 1 + t(z - 1)].$$

Line from the north pole to the finite plane Z: Note $-1 \le z \le 1$ is limited to be between the two poles. We define our line $P(t)$ to go from the North pole to the Z plane at $z = 0$. When $z = 0$, $R(t)$ becomes

$$P(t) = [tX, tY, 1 - t].$$

1.2 Restricting $[x, y, z]$ to the Riemann Sphere

To restrict the points $[x, y, z]$ to be on S we require that

$$||P(t)||^2 = t^2(X^2 + Y^2) + t^2 - 2t + 1 = 1.$$

or in terms of $|Z|$

$$||P(t)||^2 = t^2(1 + |Z|^2) - 2t + 1 = 1.$$

Solving this equation for t we have

$$t = \left\{ \frac{2}{1 + |Z|^2}, 0 \right\}.$$

The root 0 corresponds to the north pole. Thus

$$P(Z) = \frac{[2X, 2Y, |Z|^2 - 1]}{|Z|^2 + 1},$$

which is the desired Eq. 2.

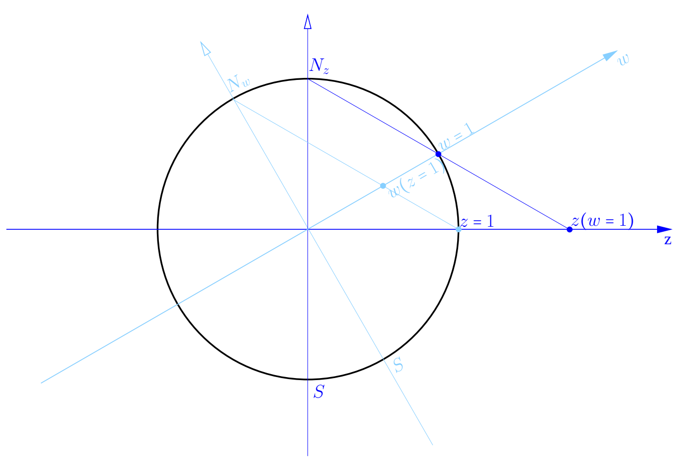

Figure 2: Superimposed mappings. The point $z = 1$ is indicated on the z axis (dark-blue) and $w = 1$ is indicated on the w axis (light blue). The projections of these points are then reflected to the other's axis. E.G., $w = 1$ is projected onto the z axis as indicated by the solid dark-blue filled circle.

2 Examples of important mappings

Here we wish to discuss some important examples, mapping out $P(Z)$ for some classic case of impedance $Z(s)$ and reflectance $\Gamma(s)$.

We begin with the item in Fig. 2 which shows two variables, z and w which are rotated by 30° relative to each other.

Some ideas

- $Z = 1/\sqrt(s)$
- The map for various bilinear transformations.

I gratefully acknowldege helpful discussions with John D'Angelo.

A Law of cotangents

For our case, ϕ is the *polar angle* and a be the length of the chord from the North Pole (N) to the puncture point α, then the triangle's sides are $a, 1, 1$. The *semi-perimeter* s is defined one-half the sum of the three sides (i.e., $s = 1 + a/2$), while the *inradius* (*the radius of the inscribed circle*)[5] is

$$r = \sqrt{\frac{(s-a)(s-1)(s-1)}{s}} = \frac{a}{2}\sqrt{\frac{a}{2+a}}. \tag{5}$$

The law of cotangents is $\cot(\phi/2) = (s-a)/r$. From Fig. 1 a is the chord form N to α.

[5]http://en.wikipedia.org/wiki/Law_of_cotangents