GREEN'S FUNCTION DERIVATIONS FOR SPECIFIC ACOUSTIC ADMITTANCES AND IMPEDANCES

BY

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2012

Urbana, Illinois

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ABSTRACT

Impedance and admittance relationships in acoustics are commonly given in their frequency-domain representations. This is done for many reasons including the simplicity of the mathematics used to compute frequency-domain impedance functions. However, although the frequency-domain representations of acoustical wave propagation typically have very neat closed-form solutions, there is a lack of intuition from the use of such techniques stemming from the added necessity of visualizing both a spatial and a frequency-domain dependence. Time-domain functions complement the frequency-domain constructs by providing new insight and intuition into important problems.

The most common geometries under investigation for acoustics are those of a propagating plane wave, an outbound spherical wave, and an outbound cylindrical wave. For all three of these geometries, there exist fully developed frequency-domain techniques to derive the corresponding impedance and admittance functions. However since any frequency-domain function must have a time-domain counterpart, there should exist time-domain representations of these functions as well.

Time-domain impedance and admittance functions for acoustical waves can be directly computed without the use of any frequency-domain methods or properties by using Green's functions. The strength of frequency-domain methods can also be realized since in simple geometries time-domain impedance functions can be easily calculated. However it is important to note that even in moderately complex geometries such as an outbound cylindrical wave, computing the time-domain impedance function can be difficult.

The end goal of the time-domain analysis of acoustic impedance and admittance functions is an improved physical understanding of acoustic wave propagation. Although frequency-domain constructs are common, they do not provide this intuition. This thesis explores derivations of time-domain

functions and provides improved intuition into these solutions.

To my wife, Puja; my parents, Ramesh and Vidya; my sister, Swathi; and all of my friends, for their love and support throughout the years

ACKNOWLEDGMENTS

First and foremost I would like to thank Dr. Jont Allen for his patience and understanding through the years. His passion for not just the search for answers but also a basic intuition into those solutions should be inspirational to all educators. Also, I would like to thank my wife, my parents, and my sister for their support through the years. Without their guidance I would not have accomplished even a fraction of what I have.

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LIST OF SYMBOLS

$\mathbf{r}$	Position vector
$\mathbf{r_0}$	Source position vector
$\mathbf{r} = (x, y, z)$	Cartesian coordinates
$\mathbf{r} = (r, \theta, \phi)$	Spherical coordinates
$\mathbf{r} = (r, \phi, z)$	Cylindrical coordinates
$p(\mathbf{r}, t)$	Acoustic pressure
$\mathbf{u}(\mathbf{r}, t)$	Particle velocity
$y(\mathbf{r}, t)$	Admittance in time-domain
$Y(\mathbf{r}, s)$	Admittance in Laplace-domain
$z(\mathbf{r}, t)$	Impedance in time-domain
$Z(\mathbf{r}, s)$	Impedance in Laplace-domain
c_0	Speed of sound
ρ_0	Density
$\delta(t)$	Dirac delta distribution
$\mathbf{1}(t)$	Unit step function
$\mathcal{G}_i(\mathbf{r}, t)$	Green's function for i-dimensional geometry
$J_\alpha(x)$	Bessel function of the first kind of the α-th order
$Y_\alpha(x)$	Bessel function of the second kind of the α-th order
$I_\alpha(x)$	Modified Bessel function of the first kind of the α-th order
$K_\alpha(x)$	Modified Bessel function of the second kind of the α-th order
$H_\alpha^{(1)}$	Hankel function of the first kind of the α-th order
$H_\alpha^{(2)}$	Hankel function of the second kind of the α-th order

CHAPTER 1

INTRODUCTION

Properties of impedance functions have been explored since the work of Schelkunoff in 1938. Since then, work has been done to better understand both the physical properties and the mathematical properties of these functions. However, impedance is mainly thought of as a frequency or Laplace-domain concept. In this domain, physical understanding of impedance has been explored to great lengths.

In a departure from most work regarding impedance functions, a bulk of the material presented here revolves around an understanding and derivation of the time-domain properties of impedance functions. The underlying theory here is that as the impedance function is a frequency-domain construct, it has a complementary, unique time-domain representation.

Better understanding of the time-domain construction of impedances is necessary for a full understanding of wave propagation. Using a frequency-domain approach to determine impedances requires first a transformation to the frequency-domain, computation to determine the frequency-domain impedance function, and then a transformation back to the time-domain. This is analogous to the approach typically seen in signal processing applications. However, just as convolution methods allow direct determination of signal systems, a full understanding of time-domain impedance functions would likewise allow easier characterization of acoustical systems.

Following, Green's functions are used to characterize the propagation of signals which obey the acoustic wave equation. The physical properties of the wave propagation can then be related to its time-domain representation and thus to its time-domain impedance function.

1.1 Scalar Linear Acoustic Wave Equation

A discussion of acoustic impedance and admittance first requires an exploration of the equations which dictate acoustic wave propagation. The main underlying equation, the acoustic wave equation, is formulated through the combination of the continuity equation (1.1) and Euler's equation (1.2) [1]. The continuity equation given as

$$-\nabla \cdot \mathbf{u}(\mathbf{r}, t) = \frac{1}{\eta P_0} \frac{\partial p(\mathbf{r}, t)}{\partial t} \tag{1.1}$$

relates particle velocity, $\mathbf{u}(\mathbf{r}, t)$, to pressure, $p(\mathbf{r}, t)$, through conservation of mass [2, p. 20] where $\mathbf{r}$ is the three-dimensional position vector. In a compressible volume of air, the total mass must remain constant [2]. Also noteworthy is that analogies between the continuity equation and Hooke's law can be made by recalling that Hooke's law states that applying a force on a system will cause an increase in potential energy in that system. Rearranging, note that $-\eta P_0 \nabla \cdot \mathbf{u}$ has units of $\frac{N}{m^2 s}$ as η is a dimensionless quantity while P_0 is the static pressure of the medium. Thus Equation (1.1) can be thought of as a density form of Hooke's law where the constants ηP_0 can be thought of as the spring constant, i.e., the stiffness of the medium, while the divergence of the particle velocity can be likened to the distance by which the air is compressed.

Euler's equation given as

$$-\nabla p(\mathbf{r}, t) = \rho_0 \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} \tag{1.2}$$

also relates particle velocity to acoustic pressure [1, pp. 117-119]. Euler's equation is analogous to the equation for force, $F = ma$, using densities instead of lumped quantities. Recalling that pressure is measured in units of $\frac{Newtons}{m^2}$, $-\nabla p$ has units of $\frac{Newtons}{m^3}$ and the density, ρ_0 is measured in units of $\frac{kg}{m^3}$. Through dimensional analysis, it can be seen that this is exactly analogous to Newton's second law of motion in terms of density.

From Euler's equation, one can also gain intuition regarding the propagation of wavefronts. As the gradient operator on a scalar function generates vectors perpendicular to the iso-contour, the velocity function generated by a pressure wave is exactly perpendicular to the iso-contours of the pressure. Thus all wave propagation occurs perpendicularly to pressure iso-contours. This facilitates the definition of the admittance and impedance functions presented in the following section.

When **r** is taken to be an argument of only one dimension, the continuity equation and Euler's equation can be rewritten in a matrix representation:

$$\frac{\partial}{\partial r}\begin{bmatrix} p(r,t) \\ u(r,t) \end{bmatrix} = -\begin{bmatrix} 0 & \rho_0 \\ \frac{1}{\eta_0 P_0} & 0 \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} p(r,t) \\ u(r,t) \end{bmatrix}. \tag{1.3}$$

This representation is known as the Webster Horn equation [3, p. 101]. Putting Equations (1.1) and (1.2) in this form helps highlight the interdependence of the pressure and velocity functions, as well as of the admittance and impedance functions as will be defined in the following section.

The continuity equation and Euler's equation can be combined to arrive at the scalar linear acoustic wave equation as given in the following [1, p. 119] where $p(\mathbf{r},t)$ is the excess pressure in units of $Pa = \frac{Newtons}{m^2}$:

$$\nabla^2 p(\mathbf{r},t) = \frac{1}{c_0^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2}. \tag{1.4}$$

This is the homogeneous acoustic scalar wave equation in free space. The homogeneous wave equation is for unforced systems, meaning this form of the wave equation is for undriven free space; i.e., there can be no impulsive inputs to the system and no continuing source: there only exists free space propagation.

The left-hand side of this equation is known as the Laplacian. The expansion of this quantity is dependent on the coordinate system being used and thus has a large influence on the solutions. These solutions can vary from being very simple in plane wave geometries to quite complex in cylindrical geometries.

By adding an additional boundary driving term, $F(\mathbf{r},t)$, the nonhomogeneous acoustic wave equation can be inferred [1, p. 140] as

$$\nabla^2 p(\mathbf{r},t) - \frac{1}{c_0^2}\frac{\partial^2 p(\mathbf{r},t)}{\partial t^2} = F(\mathbf{r},t). \tag{1.5}$$

There exist many methods to solve nonhomogeneous, driven differential equations including the following methods:

- Using linearity to decompose the equation into the homogeneous solution and the nonhomogeneous solution

- Solving by use of Fourier transforms
- Solving by use of Green's functions [4]

However for inhomogeneous partial differential equations, neither the differential equation nor the Fourier transform methods arrive easily at a general solution. The issue is that for an arbitrary input, the number of Fourier transform terms necessary may be infinite. In addition, the Fourier transform domain does not provide insight regarding the physical properties of the system and how it affects the time-domain impedance and admittance. Other solution methods may provide a better physical explanation, as will be shown in Chapters 4, 5, and 6.

1.2 Impedance and Admittance Functions

Typically *acoustic impedance* is thought of as a frequency-domain construct and is defined as the following [1, p. 286]:

$$Z(\mathbf{r}, s)\big|_{s=j\omega} \equiv \frac{P(\mathbf{r}, j\omega)}{V(\mathbf{r}, j\omega)}.$$

Here $P(\mathbf{r}, j\omega)$ is the Fourier transform of the pressure $p(\mathbf{r}, t)$ while $V(\mathbf{r}, j\omega)$ is the Fourier transform of the volume velocity at the same point. However, this definition for acoustic impedance is helpful when discussing the overall impedance, which is essentially the force divided by the speed of an entire system [1, p. 286]. This definition is helpful in characterizing an entire system such as a vibrating surface. When discussing the propagation of waves, it is more helpful to examine the *specific acoustic impedance* as defined in Equation (1.6) where now the denominator is just the particle velocity $U(\mathbf{r}, \omega)$.

$$\mathcal{Z}(\mathbf{r}, s)\big|_{s=j\omega} \equiv \frac{P(\mathbf{r}, j\omega)}{U(\mathbf{r}, j\omega)} \tag{1.6}$$

Likewise, the *specific acoustic admittance* can be defined as the reciprocal of the specific acoustic impedance as follows:

$$Y(\mathbf{r}, s)\big|_{s=j\omega} \equiv 1/Z(\mathbf{r}, s)\big|_{s=j\omega} = \frac{U(\mathbf{r}, j\omega)}{P(\mathbf{r}, j\omega)}. \tag{1.7}$$

In many cases, the admittance is more intuitive as physically one can imagine moving along a surface with constant pressure and examining how the particle velocity changes accordingly.

For a case where there is only propagation along a single direction, through the use of Euler's equation (1.2), the specific acoustic admittance can be rewritten as

$$Y(r, s) = -\frac{1}{\rho_0 s} \frac{\frac{\partial P(r,s)}{\partial r}}{P(r, s)} = -\frac{1}{\rho_0 s} \frac{\partial \log(P(r, s))}{\partial r}. \tag{1.8}$$

This form is too complex to inverse Laplace transform to yield simple time-domain solutions in all geometries therefore it can only be generically used in constructing Laplace-domain admittance functions. Yet this form highlights a few important properties of admittance functions. Primarily, it gives a very mathematically simple method of how to compute admittance functions in the Laplace-domain: the only required input is the pressure function, $P(r, s)$. Second, it makes a very strong claim that only the rate of the dropoff of the pressure affects the admittance. The rate of dropoff is a by product of the incremental change in volume of the propagating surface (this concept is highlighted by Huygen's principle in the following material). Therefore, the admittance function solutions themselves are tightly coupled with the change of area of a particular wavefront.

The definitions of admittance and impedance functions and their relations to pressure and velocity can be written succinctly by returning to Equation (1.3), taking the Laplace transform, and transforming into the following matrix form:

$$\frac{\partial}{\partial x} \begin{bmatrix} P(x, s) \\ U(x, s) \end{bmatrix} = - \begin{bmatrix} 0 & \mathcal{Z}(x, s) \\ \mathcal{Y}(x, s) & 0 \end{bmatrix} \begin{bmatrix} P(x, s) \\ U(x, s) \end{bmatrix}. \tag{1.9}$$

Here $\mathcal{Z}(x, s) = s\rho_0$ and $\mathcal{Y}(x, s) = \frac{s}{\eta P_0}$.

Using Laplace transform methods, there is an implication that a time-domain construct for impedance and admittance must also exist. Rearranging and then taking inverse Laplace transforms, the specific acoustic impedance can be written as the following convolution (see Appendix A.2):

$$p(\mathbf{r}, t) = z(\mathbf{r}, t) * u(\mathbf{r}, t), \tag{1.10}$$

and the specific acoustic admittance can be written as

$$u(\mathbf{r}, t) = y(\mathbf{r}, t) * p(\mathbf{r}, t). \tag{1.11}$$

By recalling that the admittance is the frequency-domain inverse of impedance, the following relationship for the impedance and admittance can be noted:

$$y(\mathbf{r}, t) * z(\mathbf{r}, t) = \delta(\mathbf{r}, t). \tag{1.12}$$

This result is important because it implies that in the time-domain, translation from impedance to admittance requires a deconvolution whereas in the frequency-domain, a simple mathematical inversion is required. Thus whenever a frequency-domain impedance or admittance is known, it is trivial to find the other. However, in the time-domain, this operation is not guaranteed to be mathematically simple to compute.

1.3 Reflectance

At any boundary, the characteristics of the media at that boundary point have an effect on the propagation of waves at that point. Depending on the properties of the two media, the propagating wave can be fully transferred from the first medium to the second, or it can be partially or fully reflected [1, p. 152]. The *reflectance coefficient* is a measure of the strength of a reflected wave to the strength of the incident wave at a boundary. The reflectance coefficient is defined as the following in terms of the impedances of the two media:

$$\Gamma(s) \equiv \frac{Z_1(s) - Z_2(s)}{Z_1(s) + Z_2(s)} \longleftrightarrow \gamma(t), \tag{1.13}$$

where $\gamma(t)$ is the inverse Laplace transform of $\Gamma(s)$. In terms of admittances, this can be written as

$$\Gamma(s) = \frac{\frac{1}{Y_1(s)} - \frac{1}{Y_2(s)}}{\frac{1}{Y_1(s)} + \frac{1}{Y_2(s)}} = \frac{Y_2(s) - Y_1(s)}{Y_2(s) + Y_1(s)}. \tag{1.14}$$

To simplify to the case of freewave boundary, allow $Z_2(s) = \rho_0 c_0$ to yield

$$\Gamma(s) \equiv \frac{Z_{rad}(s) - \rho_0 c_0}{Z_{rad}(s) + \rho_0 c_0} \longleftrightarrow \gamma(t). \tag{1.15}$$

The substitution of $Z_1(s) = \rho_0 c_0$ is done as it is the specific characteristic impedance of free-space and is also equivalent to the impedance of a propagating plane wave shown in later chapters. This form can be used directly to compute the reflectance of propagating acoustic waves.

1.4 Summary of Green's Function Solutions to the Wave Equation

Using the Green's function solutions as outlined in the following chapters, time-domain impedance and admittance functions can be derived directly in the time-domain. The caveat being that two-dimensional cylindrical geometry only lends itself to solutions containing infinite series for the impedance and admittance functions in the time-domain. To be explicit, the solutions to the two-dimensional geometry consist of Bessel functions in the Laplace-domain which implies complex functions in the time-domain. In addition, there are also further complexities with respect to the ability to derive integrable functions for the pressure and particle velocity such that admittance functions can then be determined. Thus there is no one, single method that can be used in all geometries; each geometry may require the use of different analytical techniques.

All of the solutions are computed by using the Green's function result of the appropriate geometry to determine a pressure function $p(\mathbf{r}, t)$ and then using Euler's equation to determine the particle velocity. From there, different mathematical techniques were used as required to deconvolve the pressure and velocity functions to determine both the admittance and the impedance functions. In the following, we summarize the main results for the one-, two-, and three-dimensional cases.

One-dimensional:

$$p(x,t) = \mathcal{G}_1(x,t) = 2c_0\pi\mathbf{1}\left(t - \frac{|x|}{c_0}\right) \longleftrightarrow P(x,s) = 2c_0\pi\frac{e^{-\frac{|x|s}{c_0}}}{s}$$

$$u(x,t) = \frac{2\pi}{\rho_0}\mathbf{1}\left(t - \frac{|x|}{c_0}\right) \longleftrightarrow U(x,s) = 2\pi\frac{e^{-\frac{|x|s}{c_0}}}{s\rho_0}$$

$$z(x,t) = \rho_0 c_0 \delta(t) \longleftrightarrow Z(x,s) = \rho_0 c_0$$

$$y(x,t) = \frac{\delta(t)}{\rho_0 c_0} \longleftrightarrow Y(x,s) = \frac{1}{\rho_0 c_0}$$

$$\gamma(x,t) = 0 \longleftrightarrow \Gamma(x,s) = 0$$

The one-dimensional results are fairly straightforward with the caveat that the pressure function, $p(x,t)$ contains a unit-step function rather than a delta function as would be expected. As highlighted in the following chapters, this is a mathematical result of the Green's function derivation for the one-dimensional geometry. This result, although not intuitive at first, can be fully explained by examining the initial-value problem for the one-dimensional case. Note that this step function "cancels out" in the impedance $z(x,t)$. Thus the admittance, impedance, and reflectance functions exactly follow the expected results.

For the two-dimensional results, the time-domain transforms for the admittance, impedance, and reflectance functions are complex and not computable as shown in the following.

Two-dimensional:

$$
\begin{aligned}
p(r,t) = \mathcal{G}_2(r,t) &= \frac{2c_0\mathbf{1}(c_0t-r)}{\sqrt{c_0^2t^2-r^2}} \longleftrightarrow P(r,s) = \frac{1}{c_0}K_0\left(\frac{sr}{c_0}\right) \\
u(r,t) = -\frac{1}{\rho_0}\frac{\partial}{\partial r}&\int_{-\infty}^{\infty}\frac{2c_0\mathbf{1}(c_0t-r)}{\sqrt{c_0^2t^2-r^2}}dt \longleftrightarrow U(r,s) = \frac{1}{\rho_0 c_0^2}K_1\left(\frac{sr}{c_0}\right) \\
&Z(r,s) = \rho_0 c_0\frac{K_0\left(\frac{sr}{c_0}\right)}{K_1\left(\frac{sr}{c_0}\right)} \\
&Y(r,s) = \frac{1}{\rho_0 c_0}\frac{K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right)} \\
&\Gamma(r,s) = \frac{K_0\left(\frac{sr}{c_0}\right) - K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right) + K_1\left(\frac{sr}{c_0}\right)}
\end{aligned}
$$

As opposed to the two-dimensional results, the three-dimensional solutions are representable in both the time and Laplace domains.

Three-dimensional:

$$
\begin{aligned}
p(r,t) = \mathcal{G}_3(r,t) &= \frac{\delta\left(\frac{r}{c_0}-t\right)}{r} \longleftrightarrow P(r,s) = \frac{e^{-\frac{sr}{c_0}}}{r} \\
u(r,t) &= \frac{\delta\left(\frac{r}{c_0}-t\right)}{r\rho_0 c_0} + \frac{\mathbf{1}\left(\frac{r}{c_0}-t\right)}{r^2\rho_0} \longleftrightarrow U(r,s) = \frac{e^{-\frac{rs}{c_0}}}{r\rho_0 c_0} + \frac{e^{-\frac{rs}{c_0}}}{sr^2\rho_0} \\
z(r,t) &= \rho_0 c_0\delta(t) - \frac{\rho_0 c_0^2}{r}e^{\frac{-c_0 t}{r}}\mathbf{1}(t) \longleftrightarrow Z(r,s) = \frac{\rho_0 c_0 rs}{sr+c_0} \\
y(r,t) &= \frac{\delta(t)}{\rho_0 c_0} + \frac{\mathbf{1}(t)}{r\rho_0} \longleftrightarrow Y(r,s) = \frac{1}{\rho_0 c_0} + \frac{1}{sr\rho_0} \\
\gamma(r,t) &= -c_0\frac{e^{-\frac{c_0 t}{2r}}}{2r}\mathbf{1}(t) \longleftrightarrow \Gamma(r,s) = -\frac{c_0}{c_0+2rs}
\end{aligned}
$$

The three-dimensional geometry follows intuition exactly where the Green's function is a single, outward propagating wave. Unlike the two-dimensional geometry, this geometry has very easy to calculate forms.

In the following chapters, $z(\mathbf{r},t)$, $y(\mathbf{r},t)$, and $\gamma(\mathbf{r},t)$ will be derived in the time-domain using the Green's function of the appropriate geometry. The Green's functions themselves are calculated beginning with the elementary point source Green's function $\mathcal{G}_3(\mathbf{r},t)$.

1.5 Huygen's Principle

Huygen's principle states that propagation of any wavefront can be determined by decomposing the wavefront into elementary point sources and summing the result from each such source [4]. At a high-level, this implies that the effect of any wavefront can be decomposed, analyzed at a more fundamental level, and then reconstructed through summations. Mathematically, this can be written as

$$f(\mathbf{r},t) = \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\nabla_0 f(\mathbf{r_0},t_0) + \frac{\mathbf{R}}{R^3} f(\mathbf{r_0},t_0) - \frac{\mathbf{R}}{cR^2}\frac{\partial}{\partial t_0} f(\mathbf{r_0},t_0)\right]_{t_0=t-R/c} \cdot \mathbf{dS_0}. \quad (1.16)$$

This equation has many important components to note. First, the surface integral is evaluated on the entire propagating wavefront which essentially sums up the contribution of all elementary points. Second, the inner expression is evaluated at $t_0 = t - R/c$ as any wavefront is caused by the wavefront existing exactly $t - R/c$ earlier in time. Third, this integral is very complex and can be very difficult to evaluate. In this thesis, the fundamental concepts of Huygen's principle are applied to construct the desired geometries. Specifically, the two- and three-dimensional geometries are composed by summing the effect of more elementary parts. The eventual goal of being able to evaluate the admittance and impedance functions for arbitrary wavefronts is difficult and beyond the scope of the material presented here. However, if Equation (1.16) could be simplified and evaluated cleanly for a number of wavefront geometries, admittance and impedance functions can be calculated using the preceding definitions after first evaluating for the pressure and velocity functions.

CHAPTER 2

GREEN'S FUNCTIONS FOR THE SCALAR WAVE EQUATION

To perform a purely time-domain analysis of acoustic impedance and admittance functions, Green's functions can be used. As opposed to differential equation methods where time-domain solutions are derived by first working in the frequency-domain, Green's functions allow the direct characterization of time and spatially varying functions. This is possible as they examine the pure time dependence of wave propagation as the computation is performed solely in the time-domain.

Intuitively, Green's functions are closely related to the idea of the impulse response in signal processing in the following regard: the Green's function in any geometry is the response of the system to an impulse of $\delta(\mathbf{r}, t)$. Therefore, solving any differential equation using Green's functions is a two-step procedure: first, find the Green's function and second, use the Green's function solution to find the general solution of the differential equation. The first step allows an impulse response characterization of the differential equation while the second step allows for the characterization of the differential equation to any input. This second step is essentially a spatial convolution of the Green's function with the initial distribution of the source conditions.

Following Morse and Feshbach [4], the most general solution can be arrived at by examining the Green's function for the inhomogeneous wave equation (1.5) for the case where $F(\mathbf{r}, t) = -4\pi q(\mathbf{r}, t)$ and $p(\mathbf{r}, t) = f(\mathbf{r}, t)$. This is a general form where $f(\mathbf{r}, t)$ can be any function which satisfies the scalar wave equation. In terms of the motion of waves and particles, $f(\mathbf{r}, t)$ can be the particle displacement, while in acoustic terms, $f(\mathbf{r}, t)$ can be taken to be the acoustic pressure. The wave equation can then be written as

$$\nabla^2 f(\mathbf{r}, t) - \frac{1}{c_0^2}\frac{\partial^2 f(\mathbf{r}, t)}{\partial t^2} = -4\pi q(\mathbf{r}, t).$$

To solve for the Green's functions, let $q(\mathbf{r},t) = \delta(\mathbf{r}-\mathbf{r_0})\delta(t-t_0)$, where $\delta(\mathbf{r}-\mathbf{r_0})$ is the three-dimensional spatial function $\delta(\mathbf{r})$. Let $\mathcal{G}_i(\mathbf{r},t)$ itself be the Green's function solution of the appropriate geometry (i.e., $i = 1,2,3$ corresponding to the one-, two-, or three-dimensional solution) for the wave equation being solved. Thus the form of the wave equation to be solved is

$$\nabla^2\mathcal{G}_i(\mathbf{r},t) - \frac{1}{c_0^2}\frac{\partial^2\mathcal{G}_i(\mathbf{r},t)}{\partial t^2} = -4\pi\delta(\mathbf{r}-\mathbf{r_0})\delta(t-t_0). \tag{2.1}$$

Then the solution to the scalar wave equation in terms of Green's functions [4, p. 837] is

$$\begin{aligned} f(\mathbf{r},t) = &\int_0^{t^+}\int_V \mathcal{G}_i(\mathbf{r},t \mid \mathbf{r_0},t_0)q(\mathbf{r_0},t_0)dV_0dt_0 \\ &+ \frac{1}{4\pi}\int_0^{t^+}\oint_S (\mathcal{G}_i\nabla_0 f(\mathbf{r},t) - f(\mathbf{r},t)\nabla_0\mathcal{G}_i)\cdot\mathbf{dS_0}dt_0 \\ &- \frac{1}{4\pi c_0^2}\int_V\left[\left(\frac{\partial\mathcal{G}_i}{\partial t_0}\right)_{t_0=0} f_0(\mathbf{r_0}) - \mathcal{G}_{i,t_0=0}\, g_0(\mathbf{r_0})\right]dV_0. \end{aligned} \tag{2.2}$$

Note that $f_0(\mathbf{r_0})$ is the initial value at $t=0$ of f, and $g_0(\mathbf{r_0})$ is the initial value of g at $t=0$ where $g = \frac{\partial f}{\partial t}$. All of the zero subscripts denote operations performed with respect to the source. For example, ∇_0 is the gradient at the location of the source at $\mathbf{r_0}$. In Cartesian coordinates, the translation between two points in space is trivial; however, for other geometries, the translation must be properly accounted for.

All three integrals are performed enveloping the source. It is important to note that all integrations over time are performed until t^+ to ensure full integration over all delta distributions specifically those occurring at t [1].

Each of the three terms in the integral denotes a physically understandable phenomenon [4, p. 837]. The first term represents the effect of the time-varying source, $q(\mathbf{r_0},t)$, and is the volume integral around the source located at $\mathbf{r_0}$ with respect to the location of the observer at $\mathbf{r}$. This volume integral is a spatial convolution across each elementary piece of the source against the Green's function of the chosen geometry. The net effect at the observation point is the sum of each elementary contribution of the source scaled by the

[1]This is similar to the Laplace transforms where 0^- is used as the lower bound to the integrand to capture delta functions at $t=0$.

Green's function of the geometry.

The second term of the integral is the sources on the boundaries. For general wave propagation with infinite boundaries, this entire integral will equal zero as it is known that acoustic pressure drops off as distance from a source increases.

Last, the third term involves the initial conditions present in the system. Chapter 3 shows how this portion of the solution can be used to specify wave propagation in a variety of geometries.

By beginning with the constructions for Green's functions in Morse and Feshbach [4], analytic forms for acoustic pressure and particle velocity in the time-domain can be derived for the cases of one-, two-, and three-dimensional wave propagation. Using these equations for the pressure and velocity, the time-domain forms for the admittance and impedance can then be derived.

There exists a great deal of flexibility in computing impedance and admittance with Green's functions. Since impedance and admittance are characteristics of wave propagation in a geometry and not related to the specific solution itself, there is no need to fully solve the scalar wave equation to determine the impedance and admittance. Once the exact Green's function for the problem is determined, it can be treated as the wave propagation function itself as it is in fact one example of an exact solution to the differential equation with an impulse as an input. This removes the requirement that the differential equation be fully solved if only the impedance or admittance functions are to be determined.

The determination of Green's function in one, two, and three dimensions begins by first characterizing an elemental source radiating in three dimensions. The other two geometries (i.e., those of one- and two-dimensional spaces) can be then deduced by summing an infinite number of elemental sources in the appropriate geometries.

2.1 Three-Dimensional Green's Function

Following the construction of Morse and Feshbach and recalling Equation (2.1), the solution to the Green's function for three-dimensional wave propagation is the following [4, p. 838]:

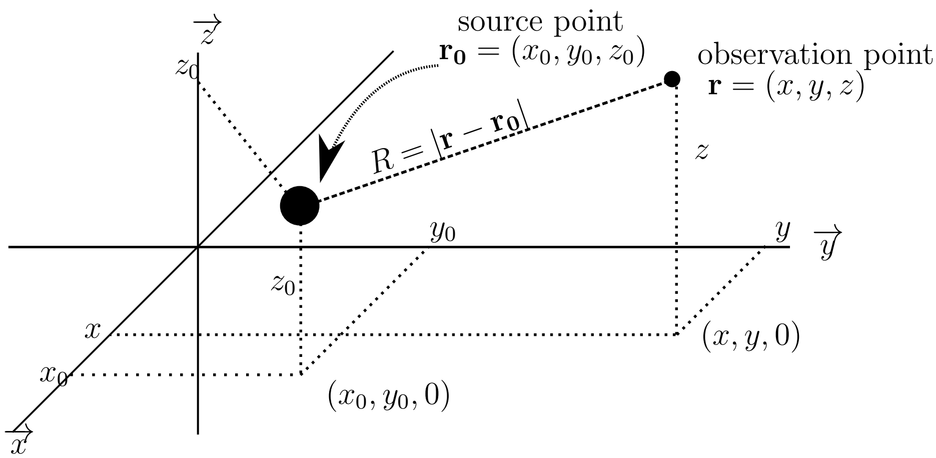

Figure 2.1: Three-dimensional Green's function

$$\mathcal{G}_3(\mathbf{r}, t|\mathbf{r_0}, t_0) = \frac{\delta\left(\frac{|\mathbf{r}-\mathbf{r_0}|}{c_0} - (t - t_0)\right)}{|\mathbf{r} - \mathbf{r_0}|}; |\mathbf{r} - \mathbf{r_0}|, t - t_0 \geq 0. \tag{2.3}$$

Per Figure 2.1, this represents the response of the system measured at $\mathbf{r}$ to a singular impulse occurring at the origin, $\mathbf{r_0}$, at time t_0 and spreading outward spherically. Thus, this is exactly the scenario of three-dimensional wave propagation.

This form can be simplified by noting the relations in Figure 2.1 that $R = |\mathbf{r} - \mathbf{r_0}|$ and by letting $\tau = t - t_0$:

$$\mathcal{G}_3(R, \tau) = \frac{\delta\left(\frac{R}{c_0} - \tau\right)}{R}; R > 0, \tau \geq 0. \tag{2.4}$$

It is good to note that as a result of the denominator, there is an implied singularity. The singularity does have a physical meaning for backward propagating waves: in that case, an inbound spherical wave will have infinite response at the location where the inbound waves converge. However, this singularity is not problematic as the function will not be evaluated at that point.

Using these basic solutions, the Green's functions for more complex geometries can be derived. The two geometries of interest here are that of a line of sources and a plane of sources. The geometry for a single line of sources is of interest as it is the geometry of an outbound traveling cylindrical wave, while the geometry of a plane of sources is the geometry for a plane progressive wave. The solutions for each of these scenarios will be constructed by using the three-dimensional solution as an elementary source and treating the other sources as infinite collections of these three-dimensional sources. This is in essence a spatial convolution of the more compact, elementary sources.

2.2 Two-Dimensional Green's Function

Two variations of the derivation for the Green's function in two dimensions are explored. First, a simplified version of Morse and Feshbach's derivation is shown. The derivation as given is complete as it allows for both the source and the observation point to be placed freely in space, however, it is complex and lacks clarity; there are no figures and several mathematical steps are omitted. Thus, a simplified version is performed first where the source is fixed at the origin starting at time $t = 0$ with only the observation point being arbitrary. As is discussed in the following section, this simplified derivation can be performed without loss of generality and with increased transparency. After the simplified derivation is examined, the full derivation by Morse and Feshbach is also shown with figures and explanatory text.

2.2.1 Simplified derivation

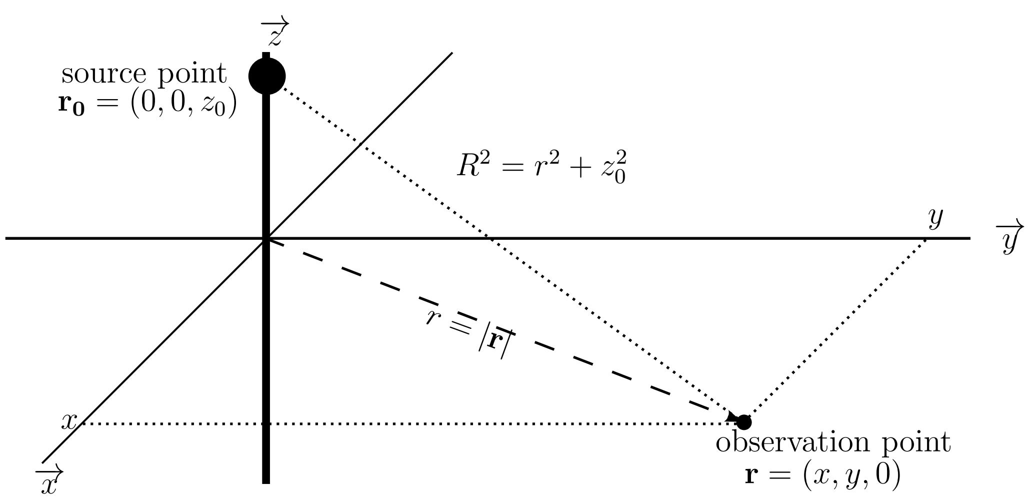

Figure 2.2: Two-dimensional Green's function

To determine the Green's function for cylindrical wave propagation, imagine sources $q(\mathbf{r_0})$ all aligned on the z-axis as shown in Figure 2.2. Each of these sources is an infinitesimal source radiating outward spherically to the surrounding medium. Thus to find the Green's function solution in two-dimensional space, we must integrate the effects of all of these elementary sources over all z_0.

Therefore to compute Green's function in two-dimensional space, $\mathcal{G}_2(\mathbf{r}, t)$, in terms of $\mathcal{G}_3(\mathbf{r}, t)$ the following integral must be evaluated:

$$\mathcal{G}_2(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathcal{G}_3(R, t) dz_0 = \int_{-\infty}^{\infty} \frac{\delta\left(\frac{R}{c_0} - t\right)}{R} dz_0.$$

Essentially, all points passing through $x = 0$ and $y = 0$ are swept along all z_0 to form the line of sources. Also, note that the beginning time is $t = 0$ and thus $\mathcal{G}_2(\mathbf{r}, t) = 0$ for $t < 0$. This integral can then be simplified by noting that the integral is an even function with respect to z_0:

$$\mathcal{G}_2(\mathbf{r}, t) = 2 \int_{0}^{\infty} \frac{\delta\left(\frac{R}{c_0} - t\right)}{R} dz_0.$$

Using $R^2 = r^2 + z_0^2$ so that $2RdR = 2z_0 dz_0$ and $\frac{dR}{z_0} = \frac{dz_0}{R}$, we can transform from dz_0 to dR:

$$\mathcal{G}_2(\mathbf{r}, t) = 2 \int_{0}^{\infty} \frac{\delta\left(\frac{R}{c_0} - t\right)}{z_0} dR = 2 \int_{0}^{\infty} \frac{\delta\left(\frac{R}{c_0} - t\right)}{\sqrt{R^2 - r^2}} dR$$

Since R and r are distances, they must be greater than or equal to zero. Then the previous integral may be simplified by noting from causality the minimum time required for wave propagation: $t = r/c_0$ where c_0 is the speed of sound. Thus for time $t < r/c_0$, the integral will be zero:

$$\mathcal{G}_2(\mathbf{r}, t) = \begin{cases} 2 \int_0^{\infty} \frac{\delta\left(\frac{R}{c_0} - t\right)}{\sqrt{R^2 - r^2}} dR & ; \ t \geq r/c_0 \\ 0 & ; \ t < r/c_0. \end{cases}$$

Simultaneously use both the translation and scaling properties of Dirac delta distributions (see Appendix A.3) and perform the integration over R resulting in the following two-dimensional Green's function:

$$\mathcal{G}_2(\mathbf{r}, t) = \begin{cases} \frac{2c_0}{\sqrt{c_0^2 \tau^2 - r^2}} & ; \ \tau \geq r/c_0 \\ 0 & ; \ \tau < r/c_0. \end{cases}$$

To bring to a more common form, factor out c_0 and recall the definition of the unit step function (Appendix A.1) to yield:

$$\mathcal{G}_2(\mathbf{r}, t) = \frac{2}{\sqrt{t^2 - r^2/c_0^2}} \mathbf{1}\left(t - \frac{r}{c_0}\right). \tag{2.5}$$

This result is the Green's function solution in two-dimensional space for acoustic wave propagation. There exist two very important items to note: first, the possibility of a singularity and, second, the presence of a "wake."

First, just as in the three-dimensional Green's function, the mathematical singularity at $t = r/c_0$ may appear problematic but the physical effects do not create a cause for concern. The singularity at the point of origin for the impulse gives an infinite value for the Green's function but in practice, the function will never be evaluated at that point: Equation (2.5) will only be evaluated when $t > 0$ and $r > 0$. In addition, the infinite value implied at the point of the singularity implies that backward propagating waves will have an infinite response at the point where they converge which is a phenomenon seen in reality.

Second, there is an implication that an impulsive line source (i.e., an exploding wire source [5]) will have a nonzero response at any point for any time $t > r/c_0$. In addition, this response is "inversely" proportional to the square root function resulting in a "wake" [4, p. 834]. However, for one-dimensional wave propagation this wake phenomenon will not be seen, as highlighted in the Section 2.3 in the computation of Green's function $\mathcal{G}_1(x, t)$.

2.2.2 General derivation

To perform the generalized derivation for the Green's function for cylindrical wave propagation, follow the geometry in Figure 2.3. In this setup the sources $q(\mathbf{r_0})$ are all parallel to the z-axis passing through x_0 and y_0. Following the simplified derivation already presented, the following integral must be evaluated:

$$\mathcal{G}_2(\mathbf{r}, t|\mathbf{r_0}, t_0) = \int_{-\infty}^{\infty} \mathcal{G}_3(R, \tau) dz_0 = \int_{-\infty}^{\infty} \frac{\delta\left(\frac{R}{c_0} - (t - t_0)\right)}{R} dz_0.$$

To integrate, the equation must be transformed from dz_0 to dR. Begin by letting $\xi = z - z_0$. This is essentially a coordinate space shift in the observer's

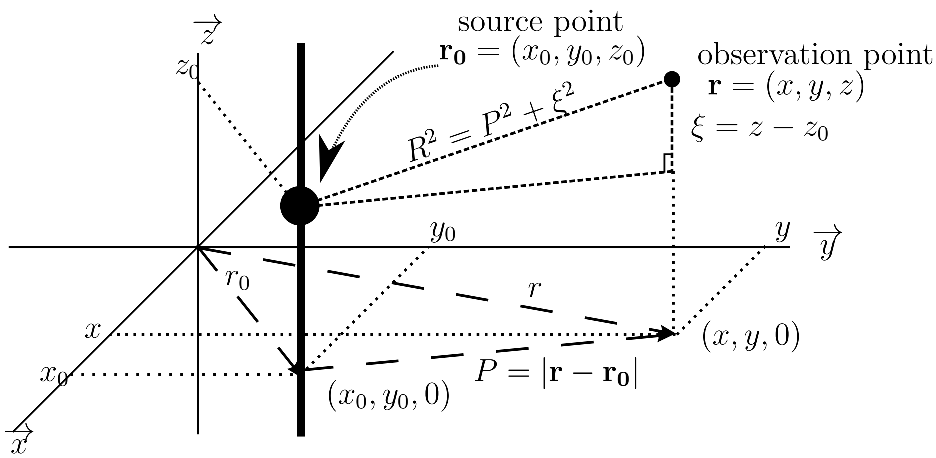

Figure 2.3: Two-dimensional Green's function

perspective along the z-axis. Since the sources are infinite in extent along the z-axis, there is no effect on the integral. The use of this translation gives $dz_0 = d\xi$. Also let $\tau = t - t_0$. The net effect of this is a simple shift in time. Last, from Figure 2.3, simplify the left-hand side arguments by recalling $P = |\mathbf{r} - \mathbf{r_0}|$. These translations transform the integral to:

$$\mathcal{G}_2(P, \tau) = \int_{-\infty}^{\infty} \frac{\delta\left(\frac{R}{c_0} - \tau\right)}{R} d\xi.$$

This integral can then be simplified by noting that the integral is an even function with respect to ξ:

$$\mathcal{G}_2(P, \tau) = 2\int_{0}^{\infty} \frac{\delta\left(\frac{R}{c_0} - \tau\right)}{R} d\xi.$$

Using $R^2 = P^2 + \xi^2$ so that $2RdR = 2\xi d\xi$ and $\frac{dR}{\xi} = \frac{d\xi}{R}$, we can transform from $d\xi$ to dR:

$$\mathcal{G}_2(P, \tau) = 2\int_{0}^{\infty} \frac{\delta\left(\frac{R}{c_0} - \tau\right)}{\xi} dR = 2\int_{0}^{\infty} \frac{\delta\left(\frac{R}{c_0} - \tau\right)}{\sqrt{R^2 - P^2}} dR.$$

Following the same minimum propagation time argument of the simplified derivation, since R and P are distances, they must be greater than or equal to zero. Thus for time $\tau < P/c_0$, the integral will be zero:

$$\mathcal{G}_2(P,\tau) = \begin{cases} 2\int_0^\infty \frac{\delta\left(\frac{R}{c_0}-\tau\right)}{\sqrt{R^2-P^2}} dR & ; \ \tau \geq P/c_0 \\ 0 & ; \ \tau < P/c_0. \end{cases}$$

Once again, using both the translation and scaling properties of Dirac delta distributions and performing the integration over R results in the following two-dimensional Green's function:

$$\begin{aligned} \mathcal{G}_2(P,\tau) &= \begin{cases} \frac{2c_0}{\sqrt{c_0^2\tau^2-P^2}} & ; \ \tau \geq P/c_0 \\ 0 & ; \ \tau < P/c_0. \end{cases} \\ &= \begin{cases} \frac{2}{\sqrt{\tau^2-P^2/c_0^2}} & ; \ \tau \geq P/c_0 \\ 0 & ; \ \tau < P/c_0 \end{cases} \\ &= \frac{2}{\sqrt{\tau^2 - P^2/c_0^2}} \mathbf{1}\left(\tau - P/c_0\right). \end{aligned}$$

After substituting back for the transformed variables, this solution is:

$$\mathcal{G}_2(\mathbf{r},t|\mathbf{r_0},t_0) = \frac{2}{\sqrt{(t-t_0)^2 - |\mathbf{r}-\mathbf{r_0}|^2/c_0^2}} \mathbf{1}\left((t-t_0) - |\mathbf{r}-\mathbf{r_0}|/c_0\right). \quad (2.6)$$

The only differences between Equations (2.5) and (2.6) are a shift in time from $\tau = t - t_0$ and a spatial shift in the observation point $|r - r_0|$, neither of which impacts the integral's evaluation.

2.3 One-Dimensional Green's Function

The one-dimensional plane wave propagation is solved by a similar method to the two-dimensional case: the more elementary source line, $\mathcal{G}_2(\mathbf{r},t)$, is superimposed to build the desired plane wave geometry, $\mathcal{G}_1(\mathbf{r},t)$. In the two-dimensional case, a two-dimensionally radiating source was formed using a collection of three-dimensionally radiating sources. In the case of one-dimensional wave propagation case, the elementary source is a collection of line sources which are then integrated over the plane, as is shown in

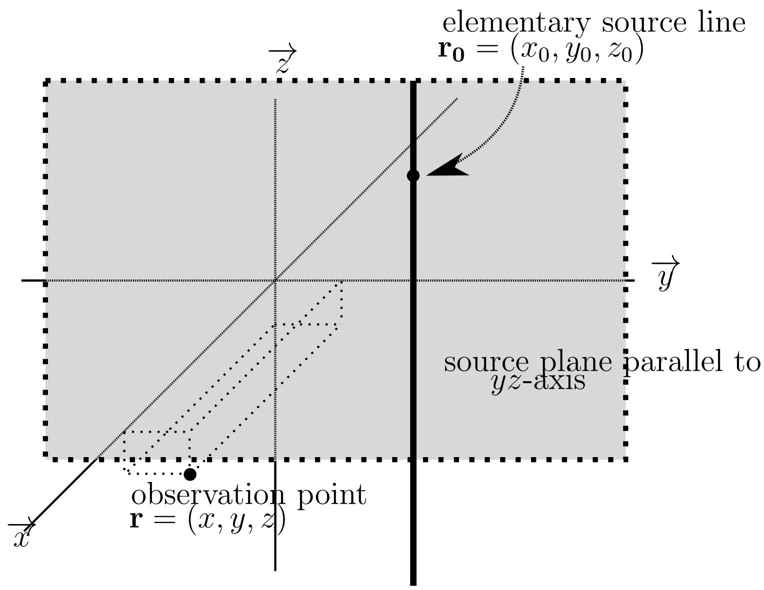

Figure 2.4: One-dimensional Green's function. The gray box represents the source plane, perpendicular to the $\hat{x}$ axis.

Figure 2.4. Similar to the two-dimensional derivation, the one-dimensional derivation here is performed for two different setups: one with a simplified geometry and one for the general case, as in Morse and Feshbach.

2.3.1 Simplified derivation

For the simplified case, follow the geometry in Figure 2.5 for which $\overrightarrow{z}$ is perpendicular to and out of the page. Compared to Figure 2.4, a couple of simplifications and translations are made. In this case, the observation point is fixed at $\mathbf{r} = (x, 0, 0)$, the source plane is at the origin $x = 0$, and $\mathcal{G}_1(\mathbf{r}, t)$ for $t < 0$. Since this geometry only has one degree of freedom, instead of requiring the three-dimensional vector $\mathbf{r}$ to denote the position of the observer, we can only use x and thus simplify $\mathcal{G}_1(\mathbf{r}, t)$ to $\mathcal{G}_1(x, t)$. The elementary sources are then line sources parallel to the z-axis, thus we must integrate over y_0:

$$\mathcal{G}_1(x, t) = \int_{-\infty}^{\infty} \mathcal{G}_2(R, t) dy_0.$$

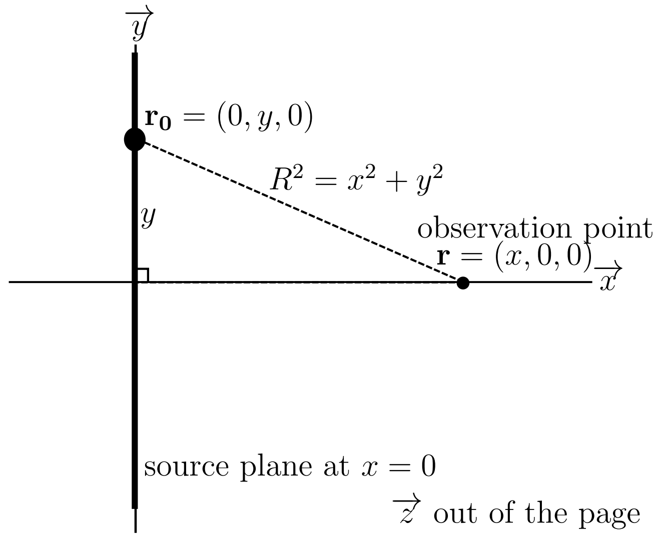

Figure 2.5: One-dimensional Green's function

Substituting the results from two-dimensional wave propagation case, Equation (2.5), this integral is

$$\mathcal{G}_1(x, t) = \int_{-\infty}^{\infty} \frac{2}{\sqrt{t^2 - R^2/c_0^2}} \mathbf{1}\left(t - \frac{R}{c_0}\right) dy_0.$$

Similar to the two-dimensional case, the variable of integration, y_0, is swept along all elementary line sources; here y_0 sweeps along the line passing through x_0 to form the source plane. By noting that from the unit step function the inner expression is only nonzero for $t \geq R/c_0 = \sqrt{x^2 + y^2}/c_0$, it can be found that $-\sqrt{c_0^2 t^2 - x^2} \leq y \leq \sqrt{c_0^2 t^2 - x^2}$.

$$\mathcal{G}_1(x, t) = 2c_0 \int_{-\sqrt{c_0^2 t^2 - x^2}}^{\sqrt{c_0^2 t^2 - x^2}} \frac{1}{\sqrt{c_0^2 t^2 - x^2 - y^2}} dy.$$

By noting that this integral is an even function, the bounds can be simplified:

$$\mathcal{G}_1(x, t) = 4c_0 \int_{0}^{\sqrt{c_0^2 t^2 - x^2}} \frac{1}{\sqrt{c_0^2 t^2 - x^2 - y^2}} dy.$$

The equation can be further simplified by applying argument regarding the minimum propagation time and thus noting that the integral is only nonzero for $t \geq |x|/c_0$. The integral can thus be split into two expressions:

$$\mathcal{G}_1(x,t) = \begin{cases} 4c_0 \int_0^{\sqrt{c_0^2 t^2 - x^2}} \frac{1}{\sqrt{c_0^2 t^2 - x^2 - y^2}} dy & ; \ t \geq |x|/c_0 \\ 0 & ; \ t < |x|/c_0. \end{cases}$$

Since $\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right)$, the previous equation becomes:

$$\mathcal{G}_1(x,t) = \begin{cases} 4c_0 \arctan\left(\frac{y}{\sqrt{c_0^2 t^2 - x^2 - y^2}}\right)\Bigg|_0^{\sqrt{c_0^2 t^2 - x^2}} & ; \ t \geq |x|/c_0 \\ 0 & ; \ t < |x|/c_0. \end{cases}$$

Evaluating the bounds yields:

$$\mathcal{G}_1(x,t) = \begin{cases} 2c_0\pi & ; \ t \geq |x|/c_0 \\ 0 & ; \ t < |x|/c_0 \end{cases} = 2c_0\pi\left[1 - \mathbf{1}\left(\frac{|x|}{c_0} - t\right)\right].$$

Combining the results yields the Green's function for a simplified one-dimensional geometry:

$$\mathcal{G}_1(x,t) = 2c_0\pi\mathbf{1}\left(t - \frac{|x|}{c_0}\right). \tag{2.7}$$

This result is interesting because the function is only nonzero for times $0 \leq t \leq |x|/c_0$. During this time period, it is a constant with value $2c_0\pi$. This agrees with intuition as for the case of one-dimensional motion by recalling that the Green's functions can be thought of as being analogous to system impulse response. In the one-dimensional case, there is an expectation of an outward movement of air and then a complementary inward movement of air of the same magnitude. This is the exact result seen here. The Green's function pulses to a constant, nonzero value then returns to zero. This is the exact shape of the particle displacement curve for one-dimensional motion.

2.3.2 General derivation

For the general case, follow the geometry in Figure 2.6 for which $\vec{z}$ is perpendicular to and out of the page. This figure shows a projection of Figure 2.4 looking directly down the z-axis. In this derivation allow the observation point to float freely at $\mathbf{r} = (x, y, z)$ and allow the yz source plane

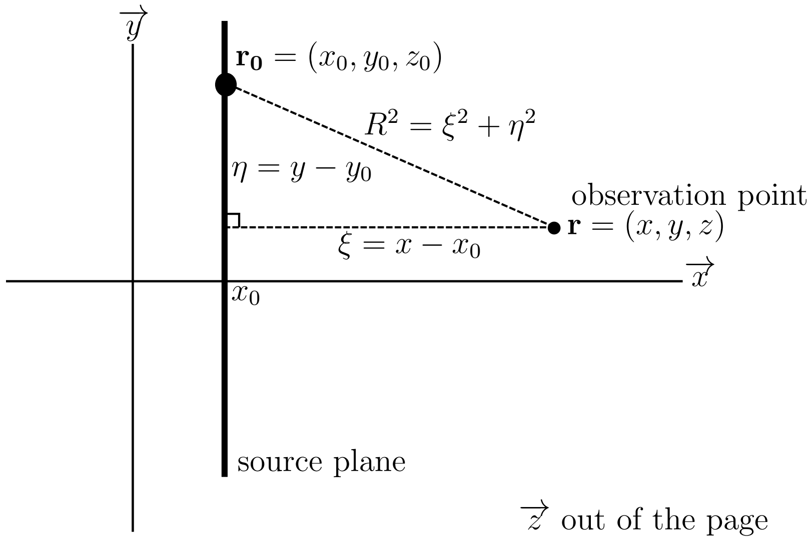

Figure 2.6: One-dimensional Green's function

to float freely at x_0. Then the integral is

$$\mathcal{G}_1(x, t|x_0, t_0) = \int_{-\infty}^{\infty} \mathcal{G}_2(R, \tau) dy_0 = \int_{-\infty}^{\infty} \frac{2}{\sqrt{\tau^2 - R^2/c_0^2}} \mathbf{1}\left(\tau - \frac{R}{c_0}\right) dR.$$

Here R is swept along all elementary line sources, thus sweeping η along the line passing through x_0 to form the source plane. By noting that from the unit step function the inner expression is only non-zero for $\tau \geq R/c_0 = \sqrt{\xi^2 + \eta^2}/c_0$. Thus $-\sqrt{c_0^2\tau^2 - \xi^2} \leq \eta \leq \sqrt{c_0^2\tau^2 - \xi^2}$.

$$\mathcal{G}_1(\xi, \tau) = 2c_0 \int_{-\sqrt{c_0^2\tau^2 - \xi^2}}^{\sqrt{c_0^2\tau^2 - \xi^2}} \frac{1}{\sqrt{c_0^2\tau^2 - \xi^2 - \eta^2}} d\eta.$$

Noting that this integral is an even function gives

$$\mathcal{G}_1(\xi, \tau) = 4c_0 \int_{0}^{\sqrt{c_0^2\tau^2 - \xi^2}} \frac{1}{\sqrt{c_0^2\tau^2 - \xi^2 - \eta^2}} d\eta.$$

Once again using the minimum propagation time argument, the integral is only non-zero for $\tau \geq |\xi| / c_0$:

$$\mathcal{G}_1(\xi, \tau) = \begin{cases} 4c_0 \int_0^{\sqrt{c_0^2\tau^2 - \xi^2}} \frac{1}{\sqrt{c_0^2\tau^2 - \xi^2 - \eta^2}} d\eta & ; \quad \tau \geq |\xi| / c_0 \\ 0 & ; \quad \tau < |\xi| / c_0. \end{cases}$$

Following the simplified derivation to compute the integral and simplify yields:

$$\mathcal{G}_1(\xi, \tau) = 2c_0\pi\mathbf{1}\left(\tau - \frac{|\xi|}{c_0}\right)$$

while substituting back for the change in variables gives the final form:

$$\mathcal{G}_1(x, t|x_0, t_0) = 2c_0\pi\mathbf{1}\left((t - t_0) - \frac{|x - x_0|}{c_0}\right), \tag{2.8}$$

Identical to the two-dimensional case, the only differences between Equations (2.7) and (2.8) are a shift in time from $\tau = t - t_0$ and a spatial shift in the observation point $|r - r_0|$, neither of which impact the integral's evaluation.

CHAPTER 3

INITIAL VALUE PROBLEM FOR THE SCALAR WAVE EQUATION

There exist two methods to determine admittance and impedance functions using Green's function. The most simple way to determine the impedance and admittance functions using Green's functions is to let the pressure function equal the Green's function. Referring to Equation (2.2) this yields the pressure function for a singular velocity impulse given at $t_0 = 0$. The second method involves using the solutions to the initial value problem which are outlined in the following. The difference between these two methods is that the first is performed by simply using the impulse response of the system to compute its characteristics while the second method uses the entire state of the system. This is done by recalling the previous discussion regarding how the solution to the initial value problem is in essence a convolution of the initial values of the system with the Green's function, therefore it is a more complete solution to the wave equation. Both methods, however, must yield identical results for the computation of the impedance and admittance functions. This is a result of the fact that the specific admittance and impedance functions are characteristics of the outbound wave and the medium and not of the state of particular wave.

Solving the initial value problem is a three-step process which begins by first determining the Green's function solution and then characterizing the initial conditions at the boundary conditions. These solutions are then input into Equation (2.2) to determine the final solution to the initial value problem.

To simplify the final solution, consider a case where there is no time-varying source; i.e. $q(\mathbf{r_0}) = 0$ allowing the first term to be dropped. This common simplification allows for the typical characterizations of the undriven, homogeneous wave equation. The second integral term can also be dropped by allowing the surface integral to be taken at an infinite distance from the source where its effect is zero. For acoustic wave propagation in free space, this agrees with physical intuition; it is known that at distances far from a sound source, i.e., at an infinite distance away, the net effects of the sound source on space are undetectable. Using these two simplifications, Equation (2.2) becomes the following:

$$f(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \int_V \left[\mathcal{G}_{i,t_0=0} g_0(\mathbf{r_0}) - \left(\frac{\partial \mathcal{G}_i}{\partial t_0} \right)_{t_0=0} f_0(\mathbf{r_0}) \right] dV_0. \tag{3.1}$$

The implications of this equation are very important. The initial value problem is a spatial convolution of two terms. One of them is that of the Green's function multiplied by the derivative of the initial conditions and the other term is that of the derivative of Green's function multiplied by the initial conditions. Both terms in this integral will have unique effects on the solutions to the initial value problem depending on the geometry.

Using Equation (3.1) and the Green's functions of the appropriate geometry as derived in the previous section, the wave equation can be solved.

3.1 One-Dimensional Initial Value Problem

Recall that in three-dimensional Cartesian geometry, an elementary unit of volume is defined as $dV_0 = dx_0 dy_0 dz_0$. However in this case, dV_0 can be simplified as the quantities in the y and z directions are all constant and thus any integration performed over those variables will cancel out and have no net effect on the solution giving $dV_0 = dx_0$. Then Equation (3.1) simplifies to

$$f(x,t) = \frac{1}{4\pi c_0^2} \int \left[\mathcal{G}_{1,t_0=0} g_0(x) - \left(\frac{\partial \mathcal{G}_1}{\partial t_0} \right)_{t_0=0} f_0(x_0) \right] dx_0. \tag{3.2}$$

To determine $\mathcal{G}_{1,t_0=0}$ and $\left(\frac{\partial \mathcal{G}_1}{\partial t_0}\right)_{t_0=0}$, substitute Equation (2.8). Thus $\mathcal{G}_{1,t_0=0} = 2c_0\pi\mathbf{1}\left(\tau - \frac{|\xi|}{c_0}\right)$ and $\left(\frac{\partial \mathcal{G}_1}{\partial t_0}\right)_{t_0=0} = -2c_0\pi\delta\left(\frac{|\xi|}{c_0} - \tau\right)$. Substitution into Equation (3.2) yields:

$$\begin{aligned} f(x,t) &= \frac{1}{4\pi c_0^2} \int \left[2c_0\pi \mathbf{1}\left(\tau - \frac{|\xi|}{c_0}\right) g_0(x) + 2c_0\pi\delta\left(\frac{|\xi|}{c_0} - \tau\right) f_0(x_0) \right] dx_0 \\ &= \frac{1}{2c_0} \int \left[\mathbf{1}\left(\tau - \frac{|\xi|}{c_0}\right) g_0(x) + \delta\left(\frac{|\xi|}{c_0} - \tau\right) f_0(x_0) \right] dx_0. \end{aligned}$$

The first term of this integral is performed by noting that the $\mathbf{1}\left(\tau - \frac{|\xi|}{c_0}\right)$ is simply the boxcar function with value equal to 1 for $c_0\tau \geq |\xi|$ and equal to 0 everywhere else. The second term is integrable using properties of the delta functions:

$$\begin{aligned} \int \delta\left(\frac{|\xi|}{c_0} - \tau\right) f_0(x_0) dx_0 &= \int \left[\delta\left(\frac{\xi}{c_0} - \tau\right) f_0(x_0) + \delta\left(\frac{-\xi}{c_0} - \tau\right) f_0(x_0) \right] dx_0 \\ &= \frac{1}{2} f_0(x + c_0 t) + \frac{1}{2} f_0(x - c_0 t). \end{aligned}$$

Combining these two results, the final solution for the wave equation in one-dimensional space is:

$$f(x,t) = \frac{1}{2} f_0(x + c_0 t) + \frac{1}{2} f_0(x - c_0 t) + \frac{1}{2c_0} \int_{x - c_0 t}^{x + c_0 t} g_0(x_0) dx_0. \qquad (3.3)$$

This solution is very similar to d'Alembert's solution to the one-dimensional wave equation [4, p. 844]. Additional insight can be had by splitting the third term of Equation (3.3) into two parts:

$$\begin{aligned} f(x,t) = \frac{1}{2} f_0(x + c_0 t) &+ \frac{1}{2c_0} \int_{x}^{x + c_0 t} g_0(x_0) dx_0 \\ &+ \frac{1}{2} f_0(x - c_0 t) + \frac{1}{2c_0} \int_{x - c_0 t}^{x} g_0(x_0) dx_0. \end{aligned}$$

Now the first two terms both have a d'Alembert-like dependency on $x + c_0 t$ while the second two terms go as $x - c_0 t$. The first and third terms are identical to what is normally seen in d'Alembert's solution and the second and fourth terms seen here are additional terms which are included due to the initial conditions of the first derivative of the propagating function. In the case where $f(x,t)$ is a function representing displacement, the first and third terms would be due to the initial conditions of the displacement while the second and fourth terms would be due to the initial conditions of the velocity function. Most texts ignore the specifics of the second and fourth term and either lump them into the general concept of the d'Alembert solution or ignore them by ignoring the effects of the initial conditions of the first derivative. This only gives a partial solution to the wave equation while the form given here is more complete.

It is interesting to note that the exact value of the function $f_0(x_0)$ is preserved over both time and space and is propagated without any scaling or other effects. This is due to the first two terms in Equation (3.3). By recalling Equation (3.2), it is interesting to note that the term including the derivative of the Green's function is the term that causes the shape of the propagating function to be preserved.

With this solution to the initial value problem, a sample pressure function for the plane wave geometry can be determined. From here, the impedance, admittance, and reflectance functions can all be calculated. For the plane wave geometry, these calculations are shown in the Chapter 4.

3.2 Two-Dimensional Initial Value Problem

To solve the initial value problem for the wave equation in two-dimensional cylindrical geometry, begin by recalling that an elementary unit of volume is given by $dV_0 = r_0 dr_0 d\phi_0 dz_0$ and an elementary unit of area is given by $dS_0 = r_0 dr_0 d\phi_0$ where $r_0 = |\mathbf{r_0}|$. Using identical arguments for eliminating variables as was used in the one-dimensional case, in this geometry the z variable can be eliminated due to it being a constant quantity in the integral yielding $dV_0 = r_0 dr_0 d\phi_0 = dS_0$. Also, without loss of generality, the observation reference point can be shifted to the origin, $r = 0$. Equation (3.1) simplifies to the following:

$$f(0,t) = \frac{1}{4\pi c_0^2} \int \left[\mathcal{G}_{2,t_0=0} g_0(\mathbf{r_0}) - \left(\frac{\partial \mathcal{G}_2}{\partial t_0} \right)_{t_0=0} f_0(\mathbf{r_0}) \right] dS_0. \tag{3.4}$$

Since the integral is performed over the entire closed surface of a cylinder at the source, the following bounds apply: $0 \leq \phi_0 \leq 2\pi$ and $0 \leq r_0 \leq c_0 t$.

$$f(0,t) = \frac{1}{4\pi c_0^2} \int_0^{2\pi} \int_0^{c_0 t} \left[\mathcal{G}_{2,t_0=0} g_0(\mathbf{r_0}) - \left(\frac{\partial \mathcal{G}_2}{\partial t_0} \right)_{t_0=0} f_0(\mathbf{r_0}) \right] r_0 dr_0 d\phi_0.$$

Substituting in for the two-dimensional Green's function, Equation (2.6):

$$f(0,t) = \frac{1}{2\pi c_0} \int_0^{2\pi} \int_0^{c_0 t} \frac{g_0(\mathbf{r_0})}{\sqrt{c_0^2 t^2 - r_0^2}} r_0 dr_0 d\phi_0 - \frac{1}{2\pi c_0} \int_0^{2\pi} \int_0^{c_0 t} \frac{\partial}{\partial t_0} \left[\frac{1}{\sqrt{c_0^2 t^2 - r_0^2}} \right] f_0(\mathbf{r_0}) r_0 dr_0 d\phi_0.$$

By noting that $\frac{\partial \mathcal{G}}{\partial t_0} = -\frac{\partial \mathcal{G}}{\partial t}$ from the fact that t and t_0 are related by $\tau = t - t_0$, we can substitute the derivative seen in the second integral above. Using this identity and switching the order of integration and differentiation for the second term yields:

$$f(0,t) = \frac{1}{2\pi c_0} \int_0^{2\pi} \int_0^{c_0 t} \frac{g_0(\mathbf{r_0})}{\sqrt{c_0^2 t^2 - r_0^2}} r_0 dr_0 d\phi_0 + \frac{\partial}{\partial t} \left[\frac{1}{2\pi c_0} \int_0^{2\pi} \int_0^{c_0 t} \frac{f_0(\mathbf{r_0})}{\sqrt{c_0^2 t^2 - r_0^2}} r_0 dr_0 d\phi_0 \right]. \quad (3.5)$$

This is the general solution of the scalar acoustic wave equation for two-dimensional wave propagation. Although the observation point was placed at the origin, i.e. that $\mathbf{r} = 0$, this simplification was done without loss of generality as the source and observation points can be shifted in space to always place the observation point at the origin. By investigating Equation (3.5) and considering the case where the initial condition $g_0(\mathbf{r_0})$ is a delta function, it is evident that as predicted by Equation (2.6), there is the presence of a wake after the initial impulse arrives; the pulse is scaled by the square root function in the denominator and is no longer a simple propagating impulse.

3.3 Three-Dimensional Initial Value Problem

To derive the complete solution to the initial value problem in three-dimensions, begin by recalling that in spherical coordinates an elementary unit of volume is given by $dV_0 = r_0^2 sin(\theta_0) d\theta_0 d\phi_0$ where once again $r_0 = |\mathbf{r_0}|$. This transforms Equation (3.1) to:

$$f(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \int\int\int \Bigg[\mathcal{G}_{3,t_0=0} g_0(\mathbf{r_0}) \\ - \left(\frac{\partial \mathcal{G}_3}{\partial t_0}\right)_{t_0=0} f_0(\mathbf{r_0}) \Bigg] r_0^2 sin(\theta_0) dr_0 d\theta_0 d\phi_0.$$

Substituting the previous result for the three-dimensional Green's function, Equation (2.3), gives:

$$f(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \int\int\int \Bigg[\frac{\delta\left(\frac{r_0}{c_0} - t\right)}{r_0} g_0(\mathbf{r_0}) \\ - \frac{\delta'\left(\frac{r_0}{c_0} - t\right)}{r_0} f_0(\mathbf{r_0}) \Bigg] r_0^2 sin(\theta_0) dr_0 d\theta_0 d\phi_0.$$

Simplifying:

$$f(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \int\int\int \Bigg[\delta\left(\frac{r_0}{c_0} - t\right) g_0(\mathbf{r_0}) \\ - \delta'\left(\frac{r_0}{c_0} - t\right) f_0(\mathbf{r_0}) \Bigg] r_0 sin(\theta_0) dr_0 d\theta_0 d\phi_0.$$

Explicitly stating that $g_0(\mathbf{r_0}) = g_0(r_0, \theta_0, \phi_0)$ and $\phi_0(\mathbf{r_0}) = \phi_0(r_0, \theta_0, \phi_0)$:

$$f(\mathbf{r},t) = \frac{1}{4\pi c_0^2} \int\int\int \Bigg[\delta\left(\frac{r_0}{c_0} - t\right) g_0(r_0, \theta_0, \phi_0) r_0 \\ - \delta'\left(\frac{r_0}{c} - t\right) f_0(r_0, \theta_0, \phi_0) r_0 \Bigg] sin(\theta_0) dr_0 d\theta_0 d\phi_0.$$

Applying differentiation properties of the delta function, (see Appendix A.3), to perform the integration over r_0 gives the final solution of the three-dimensional scalar wave equation:

$$f(\mathbf{r},t) = \frac{1}{4\pi}\int\int\left[g_0(c_0t,\theta_0,\phi_0)t + \frac{\partial}{\partial t}\left(f_0(c_0t,\theta_0,\phi_0)t\right)\right]sin(\theta_0)d\theta_0 d\phi_0 \quad (3.6)$$

This form is very powerful as there are no simplifications with respect to the geometry. However, to compute impedance and admittance functions, simplifications are made to make the source symmetric around r_0 with respect to θ_0 and ϕ_0.

Following the remarks by Morse and Feshbach [4] on page 845, it is good to note the difference between the three solutions to the scalar wave equation, Equations (3.3), (3.5), and (3.6). The one-dimensional geometry is quite simplistic and follows d'Alembert's solution directly with two terms propagating outward without a change in the shape of the initial function. To form a complete solution, there is a third term due to the direct effect of the initial conditions of the derivative of the propagating function. The two-dimensional solution is more complex as any initial conditions input into the system are smeared by the square root functions in the denominator of the solutions. This causes the presence of a wake for a simple input to the system. Lastly, the three-dimensional geometry also allows a simple propagation of the inputs, however they are scaled but not smeared. However, as opposed to the one-dimensional case, it is the derivative of the initial conditions that are propagated without any change in shape [4, p. 845].

Using these three solutions to the scalar wave equation, Equations (3.3), (3.5), and (3.6), time-domain solutions for the acoustic pressure and particle velocity can be found and thus impedance and admittance can be computed directly.

CHAPTER 4

PLANE WAVES

The case of plane wave propagation is the most simple to understand and in terms of the differential equation method, it is the most simple to solve. However, it does provide important insight by highlighting the physical differences between different forms of wave propagation and their respective admittance and impedance functions. In the plane wave case with one-dimensional wave motion, both the frequency-domain impedance and admittance are purely real. The implication of this is that there is no localized storage implied by the reflectance function.

4.1 Differential Equation Method

The differential equation method for one-dimensional wave propagation is the most simple to examine; the wave equation itself is easily separable and can then be easily analytically solved.

For the plane wave case, Equation (1.4) simplifies to the following:

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 p(x,t)}{\partial t^2}. \tag{4.1}$$

As was stated in the literature review, ignoring the effects of the backward propagating waves, the solution to this differential equation is of the following form [1, p. 122]:

$$p(x,t) = Ae^{j(\omega t - \frac{x\omega}{c_0})}.$$

Then from Equation (1.2) the particle velocity can be found to be:

$$u(x,t) = \frac{A}{\rho_0 c_0} e^{j(\omega t - \frac{x\omega}{c_0})}.$$

Thus the specific acoustic impedance and the specific acoustic admittance in the frequency-domain can be easily computed using Equations (1.6) and (1.7) as

$$Z(x,s) = \rho_0 c_0 \tag{4.2}$$

and

$$Y(x,s) = \frac{1}{\rho_0 c_0}. \tag{4.3}$$

Using Laplace transform theory the time-domain representations are respectively the following:

$$z(x,t) = \rho_0 c_0 \delta(t) \tag{4.4}$$

$$y(x,t) = \frac{1}{\rho_0 c_0}\delta(t). \tag{4.5}$$

It is important to note a few key characteristics of the specific acoustic admittance and impedance for plane waves. First, there is no imaginary term to the frequency-domain representations, so there is a very simple time-domain representation of the functions. In the time-domain, the pressure and velocity waves are just coherent scale factors of each other; there is no time delay or filtering effect.

However, the physical cause of the lack of imaginary admittance and impedance is more interesting: it strengthens the hypothesis that there is no instantaneous localized energy storage. This can be seen by computing the reflectance by recalling Equation (1.15) and substituting for $z(x,s)$. This yields:

$$\gamma(x,t) = 0 \longleftrightarrow \Gamma(x,s) = 0. \tag{4.6}$$

Thus for the case of plane wave propagation, we expect no internally reflected waves and expect all energy to be propagated directly away from the source point.

4.2 Green's Function Method

4.2.1 Characterization using only Green's function solution

It is most simple to derive the admittance and impedance functions by directly operating on the Green's function. Physically, this is valid as this is the case of simply driving the system with an impulse. For the plane wave case, begin by setting the pressure function, $p(x,t)$ equal to the Green's function solution Equation (2.8):

$$p(x,t) = \mathcal{G}_1(x,t) = 2c_0\pi\mathbf{1}\left(t - \frac{|x|}{c_0}\right) \longleftrightarrow P(x,s) = 2c_0\pi\frac{e^{-\frac{xs}{c_0}}}{s}. \tag{4.7}$$

Using Euler's equation, the particle velocity can directly be found to be the following:

$$u(x,t) = \frac{2\pi}{\rho_0}\mathbf{1}\left(t - \frac{|x|}{c_0}\right) \longleftrightarrow U(x,s) = 2\pi\frac{e^{-\frac{xs}{c_0}}}{s\rho_0}. \tag{4.8}$$

From here, it is simple to derive expressions for both the impedance and the admittance:

$$z(x,t) = \delta(t)\rho_0 c_0 \longleftrightarrow Z(x,s) = \rho_0 c_0. \tag{4.9}$$

$$y(x,t) = \frac{\delta(t)}{\rho_0 c_0} \longleftrightarrow Y(x,s) = \frac{1}{\rho_0 c_0}. \tag{4.10}$$

These solutions to the impedance and admittance functions are identical to those derived using the differential equation method outlined previously.

4.2.2 Characterization using solution to initial value problem

For completeness, these expressions can also be obtained by using the solution to the initial value problem as was stated above. Begin with Equation (3.3) where $p_0 = f(x,t)$ and $v_0 = g_0 = \frac{\partial f(x,t)}{\partial t}$.

$$p(x,t) = \frac{1}{2}p_0(x + c_0 t) + \frac{1}{2}p_0(x - c_0 t) + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} v_0(x_0)dx_0. \tag{4.11}$$

Again following the construction of Morse and Feshbach (1953a) and letting $p_0 = \delta(x)$ and $v_0 = 0$ gives

$$p(x,t) = \frac{1}{2}\delta(x - c_0 t) + \frac{1}{2}\delta(x + c_0 t). \tag{4.12}$$

From here, the use of Euler's equation directly gives the particle velocity as

$$u(x,t) = \frac{1}{2\rho_0 c_0}\delta(x - c_0 t) - \frac{1}{2\rho_0 c_0}\delta(x + c_0 t). \tag{4.13}$$

Recalling Equation (1.11), the specific acoustic admittance can easily be found to be

$$y(x,t) = \pm\frac{\delta(x)}{\rho_0 c_0} \longleftrightarrow Y(x,s) \pm \frac{1}{\rho_0 c_0}. \tag{4.14}$$

Likewise, the specific acoustic impedance can also be found:

$$z(x,t) = \pm\rho_0 c_0 \delta(x) \longleftrightarrow Z(x,s) = \pm\rho_0 c_0. \tag{4.15}$$

Here, it is important to note that the positive valued admittance and impedance functions are denoted for the forward moving wave $\delta(x - c_0 t)$ while the negative valued functions are for the backward moving wave $\delta(x + c_0 t)$. This is the exact same solution as can be found using the frequency-domain method. It is also important to note that in using the more simple method of computing the admittance and impedance just using the Green's function, two solutions were not found: only the solution for the forward moving wave was found.

CHAPTER 5

SPHERICAL WAVES

The spherical wave case is also mathematically simple as it physically is a single point source radiating in free space. Although the derivations for its acoustic impedance and admittance can be found in any acoustics text [1, 2, 6], the derivation will be reexamined here to further some points regarding time-domain impedance.

5.1 Differential Equation Method

For uniform three-dimensional radiation, through the use of spherical coordinates, the wave equation can be simplified. This can be done by noting that for symmetric spherical geometries, the Laplacian is $\nabla^2 p(r,t) = \frac{\partial^2 p(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial p(r,t)}{\partial r}$. Therefore Equation (1.4) simplifies down to a two-variable partial differential equation [1, p. 127]:

$$\frac{\partial^2 p(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial p(r,t)}{\partial r} = \frac{1}{c_0^2}\frac{\partial^2 p(r,t)}{\partial t^2}. \tag{5.1}$$

For this differential equation, the solution is a form of the d'Alembert solution to the wave equation:

$$p(r,t) = \frac{f(c_0 t - r)}{r} + \frac{g(c_0 t - r)}{r}, \tag{5.2}$$

where there is now a $1/r$ scale factor on both terms of the solution.

Then using the frequency-domain constructions given on page 127 in Kinsler, Frey, Coppens, and Sanders [1], the pressure can be represented as Equation (5.3) and the particle velocity can be represented as Equation (5.4)

$$p(r,t) = \frac{A}{r}e^{j(\omega t - \frac{r\omega}{c_0})} \tag{5.3}$$

$$u(r,t) = \frac{1}{\rho_0 c_0}\left(1 - \frac{j}{\frac{r\omega}{c_0}}\right)\frac{A}{r}e^{j(\omega t - \frac{r\omega}{c_0})}. \tag{5.4}$$

Dividing the two quantities yields the specific acoustic admittance for outbound spherical waves in the frequency-domain:

$$Y(r,j\omega) = \frac{1}{\rho_0 c_0} + \frac{1}{j\omega r \rho_0}. \tag{5.5}$$

This can also be written in the Laplace domain as:

$$Y(r,s) = \frac{1}{\rho_0 c_0} + \frac{1}{s r \rho_0}. \tag{5.6}$$

Thus either using inverse Fourier transform methods on Equation (5.5) or inverse Laplace transform methods on Equation (5.6) the time-domain admittance for spherical waves can be found to be the following:

$$y(r,t) = \frac{\delta(t)}{\rho_0 c_0} + \frac{\mathbf{1}(t)}{r\rho_0}. \tag{5.7}$$

Also, by taking the inverse of Equation (5.5) the frequency-domain representation of the specific acoustic impedance can be found to be the following:

$$Z(r,j\omega) = \rho_0 c_0 \frac{\left(\frac{r\omega}{c_0}\right)^2}{1 + \left(\frac{r\omega}{c_0}\right)^2} + j\rho_0 c_0 \frac{\frac{r\omega}{c_0}}{1 + \left(\frac{r\omega}{c_0}\right)^2}. \tag{5.8}$$

Or in the Laplace-domain:

$$Z(r,s) = \frac{\rho_0 c_0 r s}{s r + c_0} \tag{5.9}$$

$$z(r,t) = \rho_0 c_0 \delta(t) - \frac{\rho_0 c_0^2}{r} e^{\frac{-c_0 t}{r}} \mathbf{1}(t). \tag{5.10}$$

Also, the frequency-domain representation of the impedance is useful in helping to realize Equation (5.8) can be exactly represented using two parallel impedance elements: one purely real with resistance $\rho_0 c$ and one purely imaginary with reactance $r\rho_0$. As opposed to the impedance function, the dependencies here are very simple: the real term has no dependence on s or r and the imaginary term has simple dependence on s and r. The equivalent

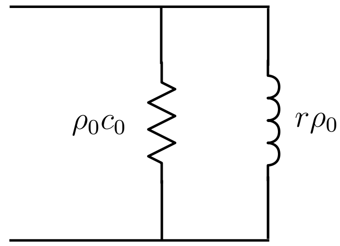

Figure 5.1: Equivalent circuit diagram for spherical propagation impedance

circuit diagram for the admittane and impedances functions is given in Figure 5.1.

To compute the reflectance, substitute Equation (5.9) into Equation (1.15) to yield the following:

$$\gamma(r,t) = -c_0 \frac{e^{-\frac{c_0 t}{2r}}}{2r} \mathbf{1}(t) \longleftrightarrow \Gamma(r,s) = -\frac{c_0}{c_0 + 2rs}. \tag{5.11}$$

The implication is that at a single localized point in space, there are internally reflected waves implied by the spherical wavefront. This can also be realized by the mechanical analogy of the impedance for three-dimensional wave propagation: the real part of the impedance acts as a resistor while the imaginary part acts as an inductor. Thus, there is local energy momentarily stored as it propagates outward.

After noting these results for the admittance and impedance, the goal is to then show that these same results can be found using Green's functions.

5.2 Green's Function Method

The simplest way to determine the impedance and admittance functions using Green's functions is to proceed as was done in the first derivation for the one-dimensional case and simply let the pressure function equal the Green's function for the three-dimensional case. This method is used here due to the complexity of the final form of the initial value problem for the three-dimensional case.

$$p(r,t) = \mathcal{G}_3(r,t) = \frac{\delta\left(\frac{r}{c_0} - t\right)}{r} \longleftrightarrow P(r,s) = \frac{e^{-\frac{sr}{c_0}}}{r}. \tag{5.12}$$

Then applying Euler's equation (1.2):

$$u(r,t) = \frac{\delta\left(\frac{r}{c_0} - t\right)}{r\rho_0 c_0} + \frac{\mathbf{1}\left(\frac{r}{c_0} - t\right)}{r^2\rho_0} \longleftrightarrow U(r,s) = \frac{e^{-\frac{rs}{c_0}}}{r\rho_0 c_0} + \frac{e^{-\frac{rs}{c_0}}}{sr^2\rho_0}. \tag{5.13}$$

From here, it is easy to derive expressions for the admittance and impedance respectively:

$$y(r,t) = \frac{\delta(t)}{\rho_0 c_0} + \frac{\mathbf{1}(t)}{r\rho_0} \longleftrightarrow Y(r,s) = \frac{1}{\rho_0 c_0} + \frac{1}{sr\rho_0} \tag{5.14}$$

$$z(r,t) = \rho_0 c_0 \delta(t) - \frac{\rho_0 c_0^2}{r} e^{\frac{-c_0 t}{r}} \mathbf{1}(t) \longleftrightarrow Z(r,s) = \frac{\rho_0 c_0 r s}{sr + c_0}. \tag{5.15}$$

This yields the same time-domain functions as derived using the frequency-domain approach. Once again, it is important to note that in this case, while it is possible to find the time-domain functions directly using Green's functions, it is not as simple as the plane wave case. This is very important as it alludes to the fact that using purely time-domain constructs for finding impedance and admittance functions may not be simple computationally.

CHAPTER 6

CYLINDRICAL WAVES

The different methods to find a closed-form solution to the impedance function for outgoing cylindrical waves encounter mathematical difficulties. The most common approach to solve the wave equation using differential equations immediately requires the use of Hankel functions to form a result for the acoustic pressure. Although this is the solution which is found in numerous sources [1, 2, 6], not a great deal of insight can be gained from the solution. Thus after a brief exploration of the properties of cylindrical wave admittance and impedance, a derivation using Green's functions is explored.

6.1 Differential Equation Method

For symmetric cylindrical geometries, the Laplacian is $\nabla^2 p(r,t) = \frac{\partial^2 p(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial p(r,t)}{\partial r}$. Thus the cylindrical wave equation simplifies to a two-variable partial differential equation [1]:

$$\frac{\partial^2 p(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial p(r,t)}{\partial r} = \frac{1}{c_0^2}\frac{\partial^2 p(r,t)}{\partial t^2}. \tag{6.1}$$

Although this equation appears very similar to Equation (5.1), the multiplicative factor of only 1 in front of the second term on the left-hand side causes a very different result: unlike the spherical or plane wave cases, there only exist transcendental solutions for the outward propagating cylindrical waves [3]. The solution must be expressed in terms of Hankel functions as given in Equation (6.2). Here $H_0^{(2)}$ is the zeroth-order Hankel function of the second kind.

$$p(r,t) = AH_0^{(2)}\left(\frac{r\omega}{c_0}\right)e^{j\omega t} \xrightarrow[\omega\to\infty]{} A\sqrt{\frac{2}{\pi\frac{r\omega}{c_0}}}e^{j(\omega t - \frac{r\omega}{c_0} + \frac{\pi}{4})}. \tag{6.2}$$

Therefore using Euler's equation, the particle velocity can be found to be the following where $H_1^{(2)}(x)$ is the first-order Hankel function of the second kind:

$$u(r,t) = -j\frac{A}{\rho_0 c_0} H_1^{(2)}\left(\frac{r\omega}{c_0}\right) e^{j\omega t}. \tag{6.3}$$

The specific acoustic admittance and the specific acoustic impedance can easily be generated using the pressure and particle velocity functions as derived earlier [1, p. 134]. These quantities are

$$Y(r,j\omega) = \frac{1}{j\rho_0 c_0} \frac{H_1^{(2)}\left(\frac{r\omega}{c_0}\right)}{H_0^{(2)}\left(\frac{r\omega}{c_0}\right)} \tag{6.4}$$

and

$$Z(r,j\omega) = j\rho_0 c_0 \frac{H_0^{(2)}\left(\frac{r\omega}{c_0}\right)}{H_1^{(2)}\left(\frac{r\omega}{c_0}\right)}. \tag{6.5}$$

These quantities can be represented in the Laplace-domain through the use of the definition of the modified Bessel function $K_\alpha(x)$ (Appendix A.4):

$$Y(r,s) = \frac{1}{\rho_0 c_0} \frac{K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right)} \tag{6.6}$$

and

$$Z(r,s) = \rho_0 c_0 \frac{K_0\left(\frac{sr}{c_0}\right)}{K_1\left(\frac{sr}{c_0}\right)}. \tag{6.7}$$

However, arriving at Laplace-domain expressions required finding the Fourier-domain representations and then inferring their Laplace-domain counterparts. In the next section, it is shown that using Green's functions, the Laplace-domain representations can be directly computed.

These two functions are very complex and little intuition can be directly gained from them. However, through expanding and simplifying in an asymptotic case, some general properties can be seen.

The reflectance can be computed directly to yield the following in the Laplace-domain:

$$\Gamma(r,s) = \frac{K_0\left(\frac{sr}{c_0}\right) - K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right) + K_1\left(\frac{sr}{c_0}\right)} \tag{6.8}$$

Unfortunately, this Laplace-transform is difficult to invert, once again showing the issues with a purely time-domain construct. $\Gamma(r,s)$ itself is an infinite series implying that the time-domain Laplace-transform pair may also be the sum of an infinite series.

Returning to the admittance function and expanding out the Hankel functions by noting that $H_n^{(2)}(x) = J_n(x) - jY_n(x)$ [1] puts Equation (6.5) into a form where it can be broken into its real and imaginary parts as follows:

$$\begin{aligned} Y(r,j\omega) &= -\frac{j}{\rho_0 c_0}\frac{H_1^{(2)}\left(\frac{r\omega}{c_0}\right)}{H_0^{(2)}\left(\frac{r\omega}{c_0}\right)} \\ &= -\frac{j}{\rho_0 c_0}\left(\frac{J_1\left(\frac{r\omega}{c_0}\right)}{J_0\left(\frac{r\omega}{c_0}\right) - jY_0\left(\frac{r\omega}{c_0}\right)} - j\frac{Y_1\left(\frac{r\omega}{c_0}\right)}{J_0\left(\frac{r\omega}{c_0}\right) - jY_0\left(\frac{r\omega}{c_0}\right)}\right) \\ &= -\frac{1}{\rho_0 c_0}\left(\frac{Y_1\left(\frac{r\omega}{c_0}\right)}{J_0\left(\frac{r\omega}{c_0}\right) - jY_0\left(\frac{r\omega}{c_0}\right)} + j\frac{J_1\left(\frac{r\omega}{c_0}\right)}{J_0\left(\frac{r\omega}{c_0}\right) - jY_0\left(\frac{r\omega}{c_0}\right)}\right) \end{aligned}$$

$$\begin{aligned} Y(r,j\omega) = -\frac{1}{\rho_0 c_0}&\frac{Y_1\left(\frac{r\omega}{c_0}\right)\left(J_0\left(\frac{r\omega}{c_0}\right) + jY_0\left(\frac{r\omega}{c_0}\right)\right)}{\left(J_0\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_0\left(\frac{r\omega}{c_0}\right)\right)^2} \\ &+ \frac{j}{\rho_0 c_0}\frac{J_1\left(\frac{r\omega}{c_0}\right)\left(J_0\left(\frac{r\omega}{c_0}\right) + jY_0\left(\frac{r\omega}{c_0}\right)\right)}{\left(J_0\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_0\left(\frac{r\omega}{c_0}\right)\right)^2} \end{aligned}$$

$$\begin{aligned} Y(r,j\omega) = -\frac{1}{\rho_0 c_0}&\frac{J_0\left(\frac{r\omega}{c_0}\right)Y_1\left(\frac{r\omega}{c_0}\right) - J_1\left(\frac{r\omega}{c_0}\right)Y_0\left(\frac{r\omega}{c_0}\right)}{\left(J_0\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_0\left(\frac{r\omega}{c_0}\right)\right)^2} \\ &- \frac{j}{\rho_0 c_0}\frac{J_1\left(\frac{r\omega}{c_0}\right)J_0\left(\frac{r\omega}{c_0}\right) + Y_1\left(\frac{r\omega}{c_0}\right)Y_0\left(\frac{r\omega}{c_0}\right)}{\left(J_0\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_0\left(\frac{r\omega}{c_0}\right)\right)^2}. \end{aligned}$$

To simplify this equation, note that the quantity $J_1\left(\frac{r\omega}{c_0}\right)Y_0\left(\frac{r\omega}{c_0}\right) - J_0\left(\frac{r\omega}{c_0}\right)Y_1\left(\frac{r\omega}{c_0}\right)$ is the Wronskian of $J_0\left(\frac{r\omega}{c_0}\right)$ and $Y_0\left(\frac{r\omega}{c_0}\right)$ and has the value $\frac{2}{\pi}\frac{c_0}{r\omega}$ [1, p. 134].

$$Y(r, j\omega) = \frac{1}{\rho_0 c_0} \frac{\frac{2}{\pi}\frac{c_0}{r\omega}}{\left(J_0\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_0\left(\frac{r\omega}{c_0}\right)\right)^2} - \frac{j}{\rho_0 c_0} \frac{J_0\left(\frac{r\omega}{c_0}\right)J_1\left(\frac{r\omega}{c_0}\right) + Y_0\left(\frac{r\omega}{c_0}\right)Y_1\left(\frac{r\omega}{c_0}\right)}{\left(J_0\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_0\left(\frac{r\omega}{c_0}\right)\right)^2}.$$

This function, although complex, has a few noteworthy relations. First, it has the typical relationship that $\lim_{\frac{r\omega}{c_0}\to\infty} Y(r, j\omega) \to \frac{1}{\rho_0 c_0}$. Thus it can be noted that as the outbound waves become more planar, they approach the same value seen for plane wave propagation.

This decomposition can be performed by a method identical to that for the impedance function:

$$Z(r, j\omega) = \rho_0 c_0 \frac{J_1\left(\frac{r\omega}{c_0}\right)Y_0\left(\frac{r\omega}{c_0}\right) - J_0\left(\frac{r\omega}{c_0}\right)Y_1\left(\frac{r\omega}{c_0}\right)}{\left(J_1\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_1\left(\frac{r\omega}{c_0}\right)\right)^2} + j\rho_0 c_0 \frac{J_0\left(\frac{r\omega}{c_0}\right)J_1\left(\frac{r\omega}{c_0}\right) + Y_0\left(\frac{r\omega}{c_0}\right)Y_1\left(\frac{r\omega}{c_0}\right)}{\left(J_1\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_1\left(\frac{r\omega}{c_0}\right)\right)^2}.$$

Then applying the value of the Wronskian as previously stated:

$$Z(r, j\omega) = \rho_0 c_0 \frac{\frac{2}{\pi}\frac{c_0}{r\omega}}{\left(J_1\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_1\left(\frac{r\omega}{c_0}\right)\right)^2} + j\rho_0 c_0 \frac{J_0\left(\frac{r\omega}{c_0}\right)J_1\left(\frac{r\omega}{c_0}\right) + Y_0\left(\frac{r\omega}{c_0}\right)Y_1\left(\frac{r\omega}{c_0}\right)}{\left(J_1\left(\frac{r\omega}{c_0}\right)\right)^2 + \left(Y_1\left(\frac{r\omega}{c_0}\right)\right)^2}. \quad (6.9)$$

Here, just as expected from examining the admittance function, the impedance function mimics the impedance of the plane wave:

$\lim_{\frac{r\omega}{c_0} \to \infty} Z(r, j\omega) \to \rho_0 c_0.$

Using the asymptotic approximations for Bessel functions found in the index, Equation (6.9) can be approximated as the following for $\frac{r\omega}{c_0} > 2$:

$$Z(r, j\omega) \approx \rho_0 c_0 + j\rho_0 c_0 \frac{J_0\left(\frac{r\omega}{c_0}\right) J_1\left(\frac{r\omega}{c_0}\right) + Y_0\left(\frac{r\omega}{c_0}\right) Y_1\left(\frac{r\omega}{c_0}\right)}{\frac{2r\omega}{c_0\pi}}. \tag{6.10}$$

This first-order approximation is helpful in defining asymptotic approximations for the time-domain impedance function for cylindrical waves.

$$z(r, t) \approx \rho_0 c_0 \delta(t) - \frac{\rho_0 c_0^2}{2r} e^{\frac{-c_0 t}{r}} \mathbf{1}(t). \tag{6.11}$$

Likewise, following a similar argument, the low frequency admittance and the time-domain admittance function can be approximated as the following:

$$Y(r, s) \approx \frac{1}{\rho_0 c_0} + \frac{1}{2rs\rho_0} \tag{6.12}$$

$$y(r, t) \approx \frac{\delta(t)}{\rho_0 c_0} + \frac{\mathbf{1}(t)}{2r\rho_0}. \tag{6.13}$$

These results are interesting as the value of the reactive term splits, i.e., is the average of, the plane wave solution and the spherical wave solution. Unfortunately, using the differential equation method, this is the best that can be done to characterize the impedance and admittance in the time-domain. Following, a derivation using purely time-domain methods is explored.

6.2 Green's Function Method

As was done in the case of one-dimensional wave propagation, the Green's function method for determining admittances and impedances is performed in two ways: one way using only the Green's function as the pressure equation and another way using the initial value problem as outlined in the previous chapters.

6.2.1 Characterization using only Green's function solution

To perform the computation using only the Green's function for two-dimensions, directly set the acoustic pressure equal to Equation (2.6). This yields:

$$p(r,t) = \mathcal{G}_2(r,t) = \frac{2c_0}{\sqrt{c_0^2t^2 - r^2}}\mathbf{1}(c_0t - r). \tag{6.14}$$

Euler's equation yields a complex equation for the particle velocity as follows: the integration with respect to time is non-trivial due to unit step function existing in the numerator while also having a $1/t$-like dependence in the denominator. This causes the integral to be difficult to perform as it is non-convergent.

$$u(r,t) = -\frac{1}{\rho_0}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\left(\frac{2c_0}{\sqrt{c_0^2t^2 - r^2}}\mathbf{1}(c_0t - r)\right)dt. \tag{6.15}$$

However, the Laplace transform for the acoustic pressure can be taken by noting $\frac{\mathbf{1}(at-x)}{\sqrt{a^2t^2-x^2}} \longleftrightarrow \frac{1}{|a|}K_0\left(\frac{sx}{a}\right)$.[1]:

$$P(r,s) = 2K_0\left(\frac{sr}{c_0}\right). \tag{6.16}$$

Due to the difficulties outlined previously with finding an expression for $u(r,t)$, $U(r,s)$ must be computed in the Laplace-domain by beginning with Equation (6.16) and by applying Laplace transform properties to Euler's equation.[2] This yields:

$$U(r,s) = \frac{2}{\rho_0c_0}K_1\left(\frac{sr}{c_0}\right). \tag{6.17}$$

Finding the admittance and impedance yields identical results to those obtained by using the differential equation method:

$$Y(r,s) = \frac{1}{\rho_0c_0}\frac{K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right)} \tag{6.18}$$

and

[1]A derivation for this Laplace transform is given in Appendix B.2.

[2]$U(r,s) = -\frac{1}{s\rho_0c_0}\frac{\partial}{\partial r}P(r,s)$.

$$Z(r,s) = \rho_0 c_0 \frac{K_0\left(\frac{sr}{c_0}\right)}{K_1\left(\frac{sr}{c_0}\right)}. \tag{6.19}$$

Here, a downfall of attempting to use purely time-domain methods can be seen: an attempt to perform all computation in the time-domain was unsuccessful. Finding an equation for the particle velocity in the Laplace-domain had to be performed on the Laplace-domain pressure equation rather than by taking the Laplace transform of the time-domain particle velocity function. However, using purely time-domain methods did facilitate the direct computation of the Laplace-domain functions without first finding their Fourier-domain counterparts and then implying the Laplace-domain functions.

6.2.2 Characterization using solution to initial value problem

To derive the admittance and impedance functions by using the Green's function solution to the initial value problem, begin with Equation (3.5) by making the simplification that $p(0,t)$ is only a function of r. That is, there is no dependence on ϕ_0, $g_0 = g_0(r_0)$ and $f_0 = f_0(r_0)$, thus Equation (3.5) simplifies to the following after performing the integration over the ϕ_0 variable:

$$p(0,t) = \frac{1}{c_0}\int_0^{c_0 t} \frac{g_0(r_0)}{\sqrt{c_0^2 t^2 - r_0^2}} r_0 dr_0 + \frac{\partial}{\partial t}\left[\frac{1}{c_0}\int_0^{c_0 t} \frac{f_0(r_0)}{\sqrt{c_0^2 t^2 - r_0^2}} r_0 dr_0\right].$$

As with the one-dimensional case, letting the initial value of the first time derivative be zero, $g_0 = 0$, and letting the initial state of the system be a simple impulse, $f_0(r_0) = \delta(r - r_0)$, results in:

$$\begin{aligned} p(0,t) &= \frac{\partial}{\partial t}\left[\frac{1}{c_0}\int_0^{c_0 t} \frac{\delta(r-r_0)}{\sqrt{c_0^2 t^2 - r_0^2}} r_0 dr_0\right] \\ &= \begin{cases} \frac{1}{c_0}\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{c_0^2 t^2 - r^2}}\right] & ; \ r < c_0\tau \\ 0 & ; \ r > c_0\tau \end{cases}. \end{aligned}$$

This can be simplified using the notation of the unit step function $\mathbf{1}(t)$ to yield the following:

$$p(0,t) = \frac{1}{c_0}\frac{\partial}{\partial t}\left[\frac{\mathbf{1}(c_0 t - r)}{\sqrt{c_0^2 t^2 - r^2}}\right] = \frac{\delta(c_0 t - r)}{\sqrt{c_0^2 t^2 - r^2}} - \frac{c_0 t \mathbf{1}(c_0 t - r)}{(c_0^2 t^2 - r^2)^{\frac{3}{2}}}. \tag{6.20}$$

Then using Euler's equation (1.2) the particle velocity can be found to be the following:

$$u(0,t) = -\frac{1}{\rho_0 c_0}\frac{\partial}{\partial r}\left[\frac{\mathbf{1}(c_0 t - r)}{\sqrt{c_0^2 t^2 - r^2}}\right] \tag{6.21}$$

$$= \frac{1}{\rho_0 c_0}\left[\frac{\delta(c_0 t - r)}{\sqrt{c_0^2 t^2 - r^2}} - \frac{\rho \mathbf{1}(c_0 t - r)}{(c_0^2 t^2 - r^2)^{\frac{3}{2}}}\right]. \tag{6.22}$$

However, this brings to light one issue regarding time-domain constructs of impedance. Deconvolution to determine acoustic admittance in two-dimensional space is not analytically possible. This can be further strengthened by noting that the admittance, Equation (6.4), is not inverse Laplace transformable. Thus in this scenario, time-domain impedance functions break down as an analysis tool.

However, a few things can be noted by comparing Equations (6.20) and (6.21). First, although the time dependence is complex, it can be noted that the time-domain impedance function will contain one term of the form $\delta(t)/\rho_0 c_0$ by noting that the first term in both Equations (6.20) and (6.21) is identical except for a scale factor. This means that, similar to the other geometries, the time-domain admittance function implies that there exists first a pulse proportional to the incoming wavefront. However, the second terms in Equations (6.20) and (6.21) highlight the complexity of the time-domain admittance function for cylindrical waves by making deconvolution impossible.

CHAPTER 7

CONCLUSIONS

Although the time-domain properties of spatial functions can provide intuition into physical effects of a problem, in some cases they are not easily computable. For example, although frequency-domain techniques are more commonly used for computing impedances and admittances for plane and spherical waves, time-domain representations also have simple transcendental solutions. However, in the case of more mathematically complex geometries such as that of a cylindrical wave, although closed-form frequency-domain representations are computable, closed-form functions for the time-domain solutions for the impedances and admittances are difficult to obtain.

Tables 7.1 - 7.3 give the analytic solutions for the admittances, impedances, and reflectance coefficients for plane and spherical wave propagation as well as approximations for the case of cylindrical waves. Recall that the cylindrical function asymptotic approximations should only be used for values of $\frac{r\omega}{c_0} > 2$ in which case they are fairly accurate.

The admittance functions are given first due to their more simple relationships. Recalling Section 5.1, the admittance functions can be represented as a combination of a resistor and an inductor in parallel. This greatly simplifies the math as there is one term that is constant, the term caused by the resistance, and only one term that is complex, the term caused by the inductance. It is interesting to note that the constant resistive term given by $\rho_0 c_0$ is the same regardless of geometry; therefore, across different geometries, only the inductive term changes. However, impedance relationships are very complex: $Z(r, s)$ has complex dependencies on both r and s. As a result of this, little intuition can be gained by direct examination of them.

Table 7.1: Admittance functions

Geometry	$y(r,t)$	$Y(r,s)$
Plane-wave	$\frac{1}{\rho_0 c_0}\delta(t)$	$\frac{1}{\rho_0 c_0}$
Cylindrical	$\approx \frac{\delta(t)}{\rho_0 c_0} + \frac{\mathbf{1}(t)}{2r\rho_0}$	$\frac{1}{\rho_0 c_0}\frac{K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right)}$
Spherical	$\frac{\delta(t)}{\rho_0 c_0} + \frac{\mathbf{1}(t)}{r\rho_0}$	$\frac{1}{\rho_0 c_0} + \frac{1}{sr\rho_0}$

Table 7.2: Impedance functions

Geometry	$z(r,t)$	$Z(r,s)$
Plane-wave	$\rho_0 c_0 \delta(t)$	$\rho_0 c_0$
Cylindrical	$\approx \rho_0 c_0 \delta(t) - \frac{\rho_0 c_0^2}{2r} e^{\frac{-c_0 t}{r}} \mathbf{1}(t)$	$\rho_0 c_0 \frac{K_0\left(\frac{sr}{c_0}\right)}{K_1\left(\frac{sr}{c_0}\right)}$
Spherical	$\rho_0 c_0 \delta(t) - \frac{\rho_0 c_0^2}{r} e^{\frac{-c_0 t}{r}} \mathbf{1}(t)$	$\frac{\rho_0 c_0 r s}{sr + c_0}$

7.1 Effects of Geometry on Impedance

The mean curvature is one measure of the total curvature of a surface. It is defined as $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ where κ_1 and κ_2 are the principal curvatures in two perpendicular directions at a point lying on a surface. The mean curvature of a plane is $H(r) = 0$, for a cylinder it is $H(r) = 1/2r$, and for a sphere it is $H(r) = 1/r$ where r here is the radius of the cylinder or of the sphere, respectively.

By noting the forms of the wave equation in Table 7.4, a few interesting claims can be made. It appears as if the mean curvature itself appears in the left-hand side of the wave equation as a part of the wave equation as two times the mean curvature for the term $\frac{\partial p(r,t)}{\partial r}$. For the different geometries, this gives very different solutions to the wave equation.

In addition, the admittance and impedance functions also show another interesting property related to the mean curvature. Defining the *impedance ratio* as $\frac{\Im(z(r,s))}{\Re(z(r,s))}$ gives a ratio of the imaginary part of the impedance function to the real part of the impedance function to. By noting the impedance functions derived in the previous chapters, Table 7.5 summarizes the impedance ratios for the geometries investigated in this thesis.

It can be seen that the impedance ratio appears to follow the values of the mean curvature alluding to fact that the curvature plays a large role in defining the properties of impedance for wave propagation. In the cylindrical case, this relationship is not very clear as it must be taken in an asymptotic

Table 7.3: Reflectance functions

Geometry	$\gamma(r,t)$	$\Gamma(r,s)$
Plane-wave	$\frac{1}{\rho_0 c_0}\delta(t)$	0
Cylindrical		$\frac{K_0\left(\frac{sr}{c_0}\right)-K_1\left(\frac{sr}{c_0}\right)}{K_0\left(\frac{sr}{c_0}\right)+K_1\left(\frac{sr}{c_0}\right)}$
Spherical	$-c_0\frac{e^{-\frac{c_0 t}{2r}}}{2r}\mathbf{1}(t)$	$-\frac{c_0}{c_0+2rs}$

Table 7.4: Wave equation for simple geometries

Geometry	$\Gamma(r,\omega)$
Plane-wave	$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 p(x,t)}{\partial t^2}$
Cylindrical	$\frac{\partial^2 p(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial p(r,t)}{\partial r} = \frac{1}{c_0^2}\frac{\partial^2 p(r,t)}{\partial t^2}$
Spherical	$\frac{\partial^2 p(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial p(r,t)}{\partial r} = \frac{1}{c_0^2}\frac{\partial^2 p(r,t)}{\partial t^2}$

case. Since the solutions to the wave equation in different geometries is not mathematically smooth with respect to the number of propagating directions, it may be difficult to do a further mathematical characterization of curvature versus impedance functions.

7.2 Summary

The use of Green's functions to directly characterize acoustic wave propagation is a powerful technique. They allow a direct, time-domain computation without the need to first perform frequency-domain transformations. In addition, the solutions themselves are intuitive as Green's functions are fundamentally an impulse response to a differential equation. In this regard, one can envision each of the Green's function solutions propagating in free space due to an impulse in the system.

However, even with relatively simple geometries, using purely time-domain solutions to the wave equation raises issues. Although the plane and spherical wave geometries are simple and directly admit solutions to the acoustic admittance and impedance, a simple geometry such as that of cylindrical waves does not.

Also, it can also be seen that the shape of the wavefront itself plays a large role in the values of admittance, impedance, and reflectance functions. A few simple cases of this phenomenon are seen in this thesis but it is believed

Table 7.5: Impedance ratios

Geometry	$\frac{\Im(z(r,s))}{\Re(z(r,s))}$
Plane-wave	0
Cylindrical	$\approx \frac{1}{2\omega c_0 r}$
Spherical	$\frac{1}{\omega c_0 r}$

that through the use of Huygen's principle, this result can be generalized.

APPENDIX A

FUNCTIONS

A.1 Properties of Unit Step Functions

Definition

$$\mathbf{1}(t) = \begin{cases} 1 & ; \quad t \geq 0 \\ 0 & ; \quad t < 0 \end{cases} \tag{A.1}$$

Relation to Dirac Delta

$$\int \delta(t - t_0)dt = \mathbf{1}(t - t_0) \tag{A.2}$$

A.2 Properties of Convolution

Definition

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau \tag{A.3}$$

A.3 Properties of Dirac Delta Distributions

Translation

$$\int_{-\infty}^{\infty} f(t)\delta(t - \tau)dt = f(\tau) \tag{A.4}$$

Scaling

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{A.5}$$

Differentiation

$$\int f(x_0)\delta'(x_0 - x)dx_0 = -f'(x) \tag{A.6}$$

Repeated Differentiation

$$\int f(t)\delta^{(n)}(t)dt = -\int \frac{\partial f}{\partial t}\delta^{(n-1)}(t)dt \tag{A.7}$$

A.4 Bessel and Hankel Functions

Bessel Functions of the First Kind

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\tfrac{1}{2}x\right)^{2m+\alpha} \tag{A.8}$$

Bessel Functions of the Second Kind

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)} \tag{A.9}$$

Hankel Functions

$$H_\alpha^{(1)}(x) = J_\alpha(x) + jY_\alpha(x) \tag{A.10}$$

$$H_\alpha^{(2)}(x) = J_\alpha(x) - jY_\alpha(x) \tag{A.11}$$

Modified Bessel Functions of the First Kind

$$I_\alpha(x) = i^{-\alpha}J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{A.12}$$

Modified Bessel Functions of the Second Kind

$$K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)} \tag{A.13}$$

$$= \frac{\pi}{2} i^{\alpha+1} H_\alpha^{(1)}(ix) = \frac{\pi}{2}(-i)^{\alpha+1} H_\alpha^{(2)}(-ix) \tag{A.14}$$

Asymptotic Approximations For values of $x \gg |\alpha^2 - 1/4|$ [1, p. 512]

$$J_\alpha(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \tag{A.15}$$

$$Y_\alpha(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \tag{A.16}$$

$$H_\alpha^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{j\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)} \tag{A.17}$$

$$H_\alpha^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-j\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)} \tag{A.18}$$

Wronskian

$$J_1(x)Y_0(x) - J_0(x)Y_1(x) = \frac{2}{\pi x} \tag{A.19}$$

APPENDIX B

LAPLACE TRANSFORMS

B.1 Properties

Differentiation

$$\frac{\partial}{\partial t}f(t) \longleftrightarrow sF(s) - f(0) \tag{B.1}$$

Integration

$$\int_0^t f(\tau)d\tau \longleftrightarrow \frac{1}{s}F(s) \tag{B.2}$$

Time Scaling

$$f(at) \longleftrightarrow \frac{1}{|a|}F\left(\frac{s}{a}\right) \tag{B.3}$$

Time Shifting

$$f(t-a)\mathbf{1}(t-a) \longleftrightarrow e^{-sa}F(s) \tag{B.4}$$

B.2 Laplace Transform Pairs

Proof $f(x,t) = \frac{\mathbf{1}(at-x)}{\sqrt{a^2t^2-x^2}} \longleftrightarrow F(x,s) = \frac{1}{|a|}K_0\left(\frac{sx}{a}\right)$.

By definition:

$$F(x,s) = \mathcal{L}\left\{f(x,t)\right\} = \mathcal{L}\left\{\frac{\mathbf{1}(at-x)}{\sqrt{a^2t^2-x^2}}\right\}.$$

To evaluate this Laplace transform, transform the variables by letting $\tau = at - x$ so that $at = \tau + x$ and $a^2t^2 = (\tau + x)^2$. Thus the argument of the square root in the denominator becomes $a^2t^2 - x^2 = (\tau + x)^2 - x^2 = \tau^2 + 2\tau x + x^2 - x^2 = \tau^2 + 2\tau x$. Then $F(x,s)$ becomes:

$$F(x,s) = \mathcal{L}\left\{\frac{\mathbf{1}(\tau)}{\sqrt{\tau^2+2\tau}}\right\}.$$

Using *Mathematica* this Laplace transform is:

$$F(x,s) = e^{sx}K_0(sx)$$

which gives the Laplace transform pair:

$$\frac{\mathbf{1}(t)}{\sqrt{t^2+2tx}} \longleftrightarrow e^{sx}K_0(sx).$$

Using the time scaling and time shifting properties of Laplace transforms, this can be transformed back into the desired form. First, using the time scaling property:

$$\frac{\mathbf{1}(at)}{\sqrt{a^2t^2+2atx}} \longleftrightarrow \frac{1}{|a|}e^{\frac{sx}{a}}K_0\left(\frac{sx}{a}\right).$$

Then applying the time shifting property:

$$\frac{\mathbf{1}\left(a(t-\frac{x}{a})\right)}{\sqrt{a^2\left(t-\frac{x}{a}\right)^2+2a^2\left(t-\frac{x}{a}\right)\frac{x}{a}}}\mathbf{1}\left(t-\frac{x}{a}\right) \longleftrightarrow \frac{1}{|a|}e^{-s\frac{x}{a}}e^{s\frac{x}{a}}K_0\left(s\frac{x}{a}\right).$$

The two unit step functions on the left-hand side are equivalent in support therefore when multiplied together, one can be dropped. Distributing the numerator and denominator on the left-hand side and simplifying the exponentionals on the right-hand side, this can be simplified to:

$$f(x,t) = \frac{\mathbf{1}(at-x)}{\sqrt{a^2t^2-x^2}} \longleftrightarrow F(x,s) = \frac{1}{|a|}K_0\left(\frac{sx}{a}\right)$$

□

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