QUARTERLY OF APPLIED MATHEMATICS
VOLUME LI, NUMBER 1
MARCH 1993, PAGES 23-35

ON HELMHOLTZ'S DECOMPOSITION THEOREM AND POISSON'S EQUATION WITH AN INFINITE DOMAIN

By

TON TRAN-CONG

Aeronautical Research Laboratory, Port Melbourne, Australia

1. Introduction. The solutions of many problems of mathematical physics depend delicately on the applicability of the classical Helmholtz [1] (also called Stokes [2]-Helmholtz) decomposition theorem. The applicability of this theorem is extended here, using a modified form of the solution to Poisson's equation, from that of the currently known version, and the theorem is also generalized into an N-dimensional version.

The theorem is known as the fundamental theorem in vector analysis (Sommerfeld [3, p. 147]) and states either that every arbitrarily given 3-dimensional vector function $\mathbf{u}(\mathbf{x})$ (subject to some condition of differentiability) can be decomposed into a curl-free vector plus another divergence-free vector (weak version), or that it can be decomposed into the gradient of a scalar function plus the curl of another vector function, i.e.,

$$\mathbf{u}(\mathbf{x}) = \nabla\theta + \nabla \times \mathbf{b} \tag{1}$$

(strong version). The theorem can be proved using the identity $\nabla^2\mathbf{w} = \nabla(\nabla\cdot\mathbf{w}) - \nabla\times(\nabla\times\mathbf{w})$ where $\mathbf{w}(\mathbf{x})$ satisfies $\nabla^2\mathbf{w} = \mathbf{u}$. However, such a simple proof, as given in vector analysis (see, e.g., Lass [4, p. 156], Aris [5, p. 70], or Bowen and Wang [6, p. 328]), requires $\mathbf{u}(\mathbf{x})$ or $\nabla\cdot\mathbf{u}$ to be of order $O(|\mathbf{x}|^{-2-\delta})$, $\delta > 0$, at infinity when the region $D \subset R^3$ under consideration is infinite. Various authors have attempted to avoid or relax such restriction. Phillips [7, p. 186], and Weatherburn [8, p. 74], used a more complicated application of the solution to Poisson's equation to relax the restriction to $|\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{-1-\delta})$, $\delta > 0$, at infinity. Blumenthal [9] devised a method of accelerating the convergence of the solution to Poisson's equation and proved that every function $\mathbf{u}(\mathbf{x}) \in C^\infty(D)$ bounded at infinity by $O(\log|\mathbf{x}|)$ can be decomposed into a curl-free vector and a divergence-free vector, which are also in $C^\infty(D)$. Gurtin [10] applied the method to prove that every $\mathbf{u}(\mathbf{x}) \in C^0(D\cup\partial D)\cap C^l(D-\partial D)$ bounded at infinity by $O(|\mathbf{x}|^{-\delta})$, $\delta > 0$, can be written as $\nabla\theta + \nabla\times\mathbf{b}$ for some $\theta(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^l(D-\partial D)$. We also have another line of approach to this decomposition problem (Nikodym [11], Friedrichs [12], Weyl [13], Bykhovski and Smirnov [14], and Fujiwara and Morimoto [15]), complementing the classical

Received May 15, 1990.
1991 *Mathematics Subject Classification.* Primary 31A30, 31B10, 53A45, 58A10.

one just mentioned, that uses functional analysis on those functions of the space L_r, $(1 < r < \infty)$. The rth powers of these functions also have to decay at infinity. Blumenthal's and Gurtin's results remain until now the least restrictive of the directly derived versions of the theorem. For examples of their use and the significance of the above discussed restriction in elasticity and fluid mechanics, see the works by Gurtin [10, 16], Millar [17], Hirasaki [18], Aregrebesola and Burley [19], Richardson and Cornish [20], and Morino [21].

Since the restriction originates from the solution to Poisson's equation with an infinite domain, it is expedient to deal with this equation, which is

$$\nabla^2\phi(\mathbf{x}) = \psi(\mathbf{x}), \tag{2}$$

where $\psi(\mathbf{x}) \in C^0(D)$, $D \subset R^N$, is an arbitrarily given function. Its classical solution is (Kellogg [22] for $N = 3$, Courant and Hilbert [23] for $N \geq 3$)

$$\phi(\mathbf{x}) = \frac{-\Gamma(N/2)}{(N-2)2\pi^{N/2}} \int_D \frac{1}{|\mathbf{y}-\mathbf{x}|^{N-2}} \psi(\mathbf{y})\, d\tau(\mathbf{y}), \qquad N \geq 3, \tag{3}$$

and is applicable to an infinite region $D \subset R^N$ only when $\psi(\mathbf{x}) = O(|\mathbf{x}|^{-2-\delta})$, $\delta > 0$, at infinity. This seemingly minor restriction is the cause for the restriction in Helmholtz's theorem and is also an obstacle to the solution of many other problems in mathematical physics (e.g., the second example of Sec. 5). This makes the relaxation of the restriction a worthwhile effort with ramifications in mathematical physics.

As Blumenthal changed the weighting factor in the integral of Eq. (3) from $1/|\mathbf{y}-\mathbf{x}|$ to $(1/|\mathbf{y}-\mathbf{x}| - 1/|\mathbf{y}|)$ so that $\psi(\mathbf{x})$ needs only be of order $O(|\mathbf{x}|^{-1-\delta})$ at infinity, we can extend his process to higher-order terms so that $\psi(\mathbf{x})$ now needs to be bounded at infinity only by $O(|\mathbf{x}|^{\alpha})$, α a constant. (Obviously, we can subsequently assume $\alpha \geq 0$ to be an integer l.) D can be simply or multiply connected. The extension is similar to the technique used in analysis to prove Carleman's inequalities (see Schechter and Simon [24], Amrein, Berthier, and Georcescu [25], and Jerison and Kenig [26]). The result here is not as general as that given in Hormander's book [27, Corollaries 10.7.10 and 10.8.2], which does not impose any limit on the growth rate of $\psi(\mathbf{x})$ and only requires D to be P-convex for singular support, but, owing to its direct derivation, should be appealing.

The solution to Poisson's equation so directly derived gives a less restricted, strong version of Helmholtz's theorem so that $\mathbf{u}(\mathbf{x})$ now needs to be bounded at infinity only by $O(|\mathbf{x}|^{l})$, l a constant (Theorem 2). D can be multiply connected and may have internal surfaces. The generalisation of Helmholtz's theorem into an N-dimensional version is realised using the calculus of differential forms. It appears that it has not been previously given as an analogue of the 3-dimensional version. (The previous tensorial work by Fosdick [28] requires that $|\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{-2-\delta})$, $\delta > 0$, at infinity and does not correspond directly to the 3-dimensional results.) Although the solution to Poisson's equation can easily give the weak version of the theorem, the strong version cannot be simply derived from the weak one via the Converse of Poincaré's Lemma unless D is topologically very simple, such as being star-shaped, which is a very restrictive situation.

Three examples are then given in Sec. 5. The first enlarges the validity of the Papkovich-Neuber general solution to Lamé's equation in linear elasticity by relaxing the requirement that $|\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{-\delta})$, $\delta > 0$, at infinity (e.g., Gurtin [10, 16] and Millar [17]). The second solves the Legendre equation. The third shows that every harmonic function can be expressed as the divergence of another harmonic vector function; the expression has been found useful in continuum mechanics (e.g., see Tran-Cong [29] and Tran-Cong and Blake [30]).

The layout of this paper is to first establish the solution to the N-dimensional Poisson's equation with an infinite domain. Helmholtz's theorem is then given with its 3-dimensional corollary. The application finally follows.

2. Definitions and notations. This paper deals with an N-dimensional space, $N \geq 3$. A letter will denote a scalar quantity when it is non-bold and an N-dimensional vector or tensor when it is latin and boldface. The letters i, j, k, l, m, n denote nonnegative integers. The symbol K denotes the set $K \equiv \{1, 2, \ldots, N-1, N\}$, $[\hat{i}] \equiv [1, 2, \ldots, \hat{i}, \ldots, N] \in K^{N-1}$ an increasing $(N-1)$-tuple with i absent from the tuple, and $[\hat{i}\hat{j}] \equiv [1, 2, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, N] \in K^{N-2}$ an increasing $(N-2)$-tuple with both i and j absent ($i < j$ always). The symbol $R \equiv (-\infty, \infty)$ denotes the set of all real numbers, $S(\mathbf{a}, \rho) \equiv \{\mathbf{x} \mid |\mathbf{x}-\mathbf{a}| < \rho\}$ an open sphere centered on $\mathbf{a}$ having radius $\rho > 0$, and $\overline{S}(\mathbf{a}, \rho) \equiv \{\mathbf{x} \mid |\mathbf{x}-\mathbf{a}| \leq \rho\}$ its corresponding closed sphere.

The permutation symbol $\delta^{12\cdots N}_{i_1 i_2 \cdots i_N}$ has the value 1, -1, or 0 depending on whether $(i_1, i_2, \ldots, i_N)$ is an even, odd, or not a permutation of $(1, 2, \ldots, N)$. The Kronecker symbol δ^j_i has the value of unity when $i = j$ and zero otherwise. $\nabla_{\mathbf{x}}$ and $d\mathbf{x}$ denote $\mathbf{e}_1 \partial/\partial x_1 + \mathbf{e}_2 \partial/\partial x_2 + \cdots + \mathbf{e}_N \partial/\partial x_N$ and $\mathbf{e}_1\, dx_1 + \mathbf{e}_2\, dx_2 + \cdots + \mathbf{e}_N\, dx_N$, respectively, where $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N)$ are the basis of the coordinate system $(x_1, x_2, \ldots, x_N)$. When no confusion can arise, the subscript of ∇ may be omitted. With a k-tuple $\mathbf{i} \equiv (i_1, i_2, \ldots, i_k)$ attached to $\mathbf{x}$, we define $d\mathbf{x}_{\mathbf{i}} \equiv dx_{i_1}\mathbf{e}_{i_1} \wedge dx_{i_2}\mathbf{e}_{i_2} \wedge \cdots \wedge dx_{i_k}\mathbf{e}_{i_k} = dx_{i_1}\, dx_{i_2} \cdots dx_{i_k}\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k}$, where $\wedge$ denotes the exterior product (Bowen and Wang [6, p. 303]).

Let $f(\mathbf{u})$ be a *scalar function* of a vectorial variable $\mathbf{u} \equiv (u_1, u_2, \ldots, u_N)$. Its differential $d^k f(\mathbf{a}, \mathbf{b})$, where $k \geq 0$, is a function defined by

$$d^k f(\mathbf{a}, \mathbf{b}) \equiv \sum_{1 \leq i_1, i_2, \ldots, i_k \leq N} \left(\frac{\partial^k f(\mathbf{u})}{\partial u_{i_1} \partial u_{i_2} \cdots \partial u_{i_k}} \right)_{\mathbf{u}=\mathbf{a}} b_{i_1} b_{i_2} \cdots b_{i_k}, \qquad k \geq 0.$$

When $\mathbf{c}(\mathbf{x})$ is a *tensor*, $d\mathbf{c}$ denotes its *exterior derivative* (Bowen and Wang [6, p. 303]); this should not be confused with the notation for the differentials.

The constant κ and functions $h(\mathbf{x})$ and $g(i, \mathbf{y}, -\mathbf{x})$, $i \geq 0$, denote

$$\kappa \equiv \Gamma(N/2)[(N-2)2\pi^{N/2}]^{-1} \tag{4}$$

and

$$h(\mathbf{x}) \equiv \kappa / |\mathbf{x}|^{N-2}, \qquad N \geq 3, \tag{5}$$

and

$$g(i, \mathbf{y}, -\mathbf{x}) \equiv h(\mathbf{y}-\mathbf{x}) - \sum_{k=0}^{i} \frac{1}{k!} d^k h(\mathbf{y}, -\mathbf{x}), \tag{6}$$

respectively, where $\Gamma(x)$ is the gamma function with argument x.

DEFINITION 1. A *domain* is an open set, any two of whose points can be joined by a polygonal line, of a finite number of segments, all of whose points belong to the set (Kellogg [22, p. 93]). A *region* D is either a domain, or a domain together with some or all of its boundary points (Kellogg [22, p. 93]). The term *regular region* is defined as in Kellogg's book [22].

DEFINITION 2. A function $f(\mathbf{x})$ defined in D satisfies a *Hölder condition* with exponent α $(\alpha > 0)$ at a point $\mathbf{a}$ if there are two positive constants $c(\mathbf{a})$ and $M(\mathbf{a})$ such that $|f(\mathbf{x}) - f(\mathbf{a})| \leq M(\mathbf{a})|\mathbf{x}-\mathbf{a}|^{\alpha}$ for any $\mathbf{x}$ such that $|\mathbf{x}-\mathbf{a}| \leq c(\mathbf{a})$.

∂D and $D - \partial D$ denote the boundary and the interior of D, respectively. $f(\mathbf{x}) \in C^n(D)$ denotes that the function $f(\mathbf{x})$ is defined, continuous together with all of its partial derivatives of order up to and including n $(n \geq 0)$ in the region D, $f(\mathbf{x}) \in C^{n,\alpha}(D)$ denotes that $f(\mathbf{x}) \in C^n(D)$ and all of its nth order partial derivatives satisfy the Hölder condition with exponent α in D.

Every kth-order $(0 \leq k \leq N)$ antisymmetric tensor $\mathbf{b}(\mathbf{x})$, which changes sign when any two of its indices are interchanged, corresponds to a differential k-form $\mathbf{b}(\mathbf{x})$ defined by (Bowen and Wang [6, p. 303])

$$\begin{aligned}\mathbf{b}(\mathbf{x}) &\equiv \sum_{1\leq i_1<i_2<\cdots<i_k\leq N} b_{i_1 i_2\cdots i_k}\, dx_{i_1}\mathbf{e}_{i_1} \wedge dx_{i_2}\mathbf{e}_{i_2} \wedge \cdots \wedge dx_{i_k}\mathbf{e}_{i_k} \\ &= \sum_{1\leq i_1<i_2<\cdots<i_k\leq N} b_{i_1 i_2\cdots i_k}\, d\mathbf{x}_{i_1 i_2\cdots i_k}.\end{aligned}$$

The symbol $(\mathbf{b})_{i_1 i_2\cdots i_k}$, $1 \leq i_1 < i_2 < \cdots < i_k \leq N$, denotes the component $b_{i_1 i_2\cdots i_k}$ of the antisymmetric tensor and the differential k-form $\mathbf{b}$.

DEFINITION 3. A differential k-form $\mathbf{b}$ $(1 \leq k \leq N)$ is *exact* if it is equal to the differential of another differential $(k-1)$-form $\mathbf{c}$, i.e., $\mathbf{b} = d\mathbf{c}$.

3. Solution to Poisson equation in an infinite domain. We first note that there is a constant $c > 0$ such that the function $u(x)$ defined by

$$u(x) \equiv \begin{cases} 1 & \text{for } x \geq \frac{1}{2}, \\ 0 & \text{for } x \leq -\frac{1}{2}, \\ c\int_{-1/2}^{x} \exp(1/(4t^2-1))\,dt & \text{otherwise} \end{cases} \tag{7}$$

is in $C^{\infty}(R)$. If $\psi(x) \in C^{n,\alpha}(D)$, $0 < \alpha < 1$, then we have $[u(x)\psi(x)] \in C^{n,\alpha}(D)$.

We next consider the Laplacian of the $(k+2)$th differential of $f(\mathbf{y})$:

$$\begin{aligned}\nabla^2_{\mathbf{x}}(d^{k+2}f(\mathbf{y},\mathbf{x})) &= \sum_{i,j\in K} \frac{\partial^2}{(\partial x_i)^2}\left[x_j \frac{\partial(d^{k+1}f(\mathbf{y},\mathbf{x}))}{\partial y_j}\right] \\ &= 2\sum_{i,j\in K} \delta_i^j \frac{\partial^2(d^{k+1}f(\mathbf{y},\mathbf{x}))}{\partial x_i \partial y_j} + \sum_{i,j\in K} x_j \frac{\partial^3(d^{k+1}f(\mathbf{y},\mathbf{x}))}{\partial y_j \partial x_i \partial x_i} \\ &= 2\sum_{i=1}^{N} \frac{\partial^2(d^{k+1}f(\mathbf{y},\mathbf{x}))}{\partial x_i \partial y_i} + \sum_{i=1}^{N} x_i \frac{\partial}{\partial y_i}[\nabla^2_{\mathbf{x}}(d^{k+1}f(\mathbf{y},\mathbf{x}))].\end{aligned}$$

But

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i}\frac{\partial}{\partial x_i}(d^{k+1} f(\mathbf{y}, \mathbf{x}))$$
$$= \sum_{1\le i, i_1, i_2, \dots, i_{k+1}\le N} \frac{\partial}{\partial y_i}\left[\frac{\partial^{k+1} f(\mathbf{y})}{\partial y_{i_1}\partial y_{i_2}\cdots\partial y_{i_{k+1}}}\frac{\partial}{\partial x_i}(x_{i_1}x_{i_2}\cdots x_{i_{k+1}})\right]$$
$$= \sum_{1\le i, i_1, i_2, \dots, i_{k+1}\le N} \frac{\partial}{\partial y_i}\left[(k+1)\left(\frac{\partial^{k+1} f(\mathbf{y})}{\partial y_{i_1}\partial y_{i_2}\cdots\partial y_{i_{k+1}}}\right)x_{i_1}x_{i_2}\cdots x_{i_k}\delta_i^{i_{k+1}}\right]$$
$$= (k+1)\nabla^2_{\mathbf{y}}(d^k f(\mathbf{y}, \mathbf{x})).$$

Hence

$$\nabla^2_{\mathbf{x}}(d^{k+2} f(\mathbf{y}, \mathbf{x})) = 2(k+1)\nabla^2_{\mathbf{y}}(d^k f(\mathbf{y}, \mathbf{x})) + \sum_{i=1}^{N} x_i \frac{\partial}{\partial y_i}(\nabla^2_{\mathbf{x}} d^{k+1} f(\mathbf{y}, \mathbf{x})), \quad (8)$$

and we have

Lemma 1. Let $h(\mathbf{y}) = \kappa/|\mathbf{y}|^{N-2}$. We have

$$\nabla^2_{\mathbf{x}}(d^k h(\mathbf{y}, \mathbf{x})) = 0 \quad \text{for } |\mathbf{y}| \ne 0 \text{ and } k \ge 0. \quad (9)$$

Proof. Direct differentiation proves the case for $k = 0$ and $k = 1$. We next note that

$$\nabla^2_{\mathbf{y}}(d^k h(\mathbf{y}, \mathbf{x})) = \sum_{1\le i_1, i_2, \dots, i_k\le N} \frac{\partial^k}{\partial y_{i_1}\partial y_{i_2}\cdots\partial y_{i_k}}\left[\nabla^2_{\mathbf{y}}\left(\frac{\kappa}{|\mathbf{y}|^{N-2}}\right)\right]x_{i_1}x_{i_2}\cdots x_{i_k} = 0 \quad \text{for } |\mathbf{y}| \ne 0$$

for all $k \ge 0$. Therefore Eq. (8) gives

$$\nabla^2_{\mathbf{x}}(d^{k+2} h(\mathbf{y}, \mathbf{x})) = \sum_{i=1}^{N} x_i \frac{\partial}{\partial y_i}(\nabla^2_{\mathbf{x}} d^{k+1} h(\mathbf{y}, \mathbf{x})),$$

whose right-hand side vanishes for $k = 0$. An induction is then applied on this to prove Eq. (9) for all values of $k \ge 1$.

Lemma 2. Let the function $\psi(\mathbf{x}) \in C^0(D \cup \partial D)$ be given for the bounded, regular region $D \subset R^N$. The potential function

$$\phi(\mathbf{x}) \equiv -\kappa \int_D \frac{\psi(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{N-2}}\, d\tau(y) \quad (10)$$

is in $C^1(D \cup \partial D)$ and satisfies

$$\nabla_{\mathbf{x}}\phi(\mathbf{x}) = -\kappa \int_D \psi(\mathbf{y})\nabla_{\mathbf{x}}\left(\frac{1}{|\mathbf{y}-\mathbf{x}|^{N-2}}\right) d\tau(\mathbf{y}) \quad (11)$$

in $D \cup \partial D$. If we further have $\psi(\mathbf{x}) \in C^{0,\alpha}(D - \partial D)$ then $\phi(\mathbf{x}) \in C^2(D - \partial D)$ and $\phi(\mathbf{x})$ satisfies in $D - \partial D$

$$\frac{\partial^2 \phi(\mathbf{x})}{\partial x_k \partial x_j} = \frac{\delta_k^j}{N}\psi(\mathbf{x}) - \kappa \lim_{\sigma \to 0} \int_{D - S(\mathbf{x}, \sigma)} \psi(\mathbf{y}) \frac{\partial^2}{\partial x_k \partial x_j} \left(\frac{1}{|\mathbf{y} - \mathbf{x}|^{N-2}} \right) d\tau(\mathbf{y}), \quad 1 \le k, j \le N, \tag{12}$$

and

$$\nabla^2 \phi = \psi. \tag{13}$$

For every $S(\mathbf{q}, \eta) \subset (D - \partial D)$ and an integer $n \ge 0$ it satisfies

$$\phi(\mathbf{x}) \in C^{n+2}(S(\mathbf{q}, \eta)) \quad \text{if } \psi(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)). \tag{14}$$

This is an N-dimensional version of the results given in the books by Schmidt [31] or Kellogg [22]; for its proof see Courant and Hilbert [23], Gilbarg and Trudinger [32], Mikhlin [33], Calderon and Zygmund [34], and Taibleson [35].

We are now ready to consider the first theorem of this paper.

THEOREM 1. Let $D \subset R^N$ be a regular region and let

$$\psi(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{0,\alpha}(D - \partial D)$$

be bounded at infinity by $O(|\mathbf{x}|^l)$ for some constant l. The potential function

$$\phi(\mathbf{x}) \equiv -\int_D \{h(\mathbf{y} - \mathbf{x})[1 - u(|\mathbf{y}| - 2)] + g(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}| - 2)\}\psi(\mathbf{y})\, d\tau(\mathbf{y}) \tag{15}$$

is in $C^1(D \cup \partial D)$ and satisfies

$$\nabla_{\mathbf{x}}\phi(\mathbf{x}) = -\int_D \psi(\mathbf{y})\nabla_{\mathbf{x}}\{h(\mathbf{y}-\mathbf{x})[1-u(|\mathbf{y}|-2)] + g(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}|-2)\}\, d\tau(\mathbf{y}) \tag{16}$$

in $D \cup \partial D$. It is also twice differentiable and satisfies in $D - \partial D$

$$\frac{\partial^2}{\partial x_i \partial x_j}\phi(\mathbf{x}) = \frac{\delta_i^j}{N}\psi(\mathbf{x}) - \lim_{\sigma \to 0} \int_{D - S(\mathbf{x}, \sigma)} \psi(\mathbf{y}) \frac{\partial^2}{\partial x_i \partial x_j}\{h(\mathbf{y} - \mathbf{x})[1 - u(|\mathbf{y}| - 2)] + g(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}| - 2)\}\, d\tau(\mathbf{y}) \qquad 1 \le i, j \le N, \tag{17}$$

and

$$\nabla^2 \phi = \psi. \tag{18}$$

For every $S(\mathbf{q}, \eta) \subset (D - \partial D)$ and any integer $n \ge 0$ it satisfies

$$\phi(\mathbf{x}) \in C^{n+2}(S(\mathbf{q}, \eta)) \quad \text{if } \psi(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)). \tag{19}$$

Proof. Let $D_1 \equiv D \cap S(\mathbf{0}, 3)$, $D_2 \equiv D - S(\mathbf{0}, 1)$, and $\psi(\mathbf{x}) = \psi_1(\mathbf{x}) + \psi_2(\mathbf{x})$ where $\psi_1(\mathbf{x}) = [1 - u(|\mathbf{x}| - 2)]\psi(\mathbf{x})$, $\psi_2(\mathbf{x}) = u(|\mathbf{x}| - 2)]\psi(\mathbf{x})$, and $\psi_1(\mathbf{x}), \psi_2(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{0,\alpha}(D - \partial D)$.

We note that $d^k h(\mathbf{y}, -\mathbf{x}) \in C^\infty(D^2 \times R^N)$ for all $k \ge 0$, that $\psi_1(\mathbf{x}) \equiv 0$ outside $D_1 \cap S(\mathbf{0}, 5/2)$, and that $\psi_2(\mathbf{x}) \equiv 0$ outside $D_2 - S(\mathbf{0}, 3/2)$. We then have $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x})$ where

$$\phi_1(\mathbf{x}) = -\int_{D_1} h(\mathbf{y} - \mathbf{x})\psi_1(\mathbf{y})\, d\tau(\mathbf{y}) \quad \text{and} \quad \phi_2(\mathbf{x}) = -\int_{D_2} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}).$$

Since D_1 is finite, only $\phi_2(\mathbf{x})$ needs to be considered. Taylor's theorem gives $g(l+2, \mathbf{y}, -\mathbf{x}) = (d^{l+3}h(\mathbf{y}-t\mathbf{x}, -\mathbf{x}))/(l+3)!$ where $0 < t < 1$. As $|\mathbf{y}| \to \infty$, $g(l+2, \mathbf{y}, -\mathbf{x}) = O(|\mathbf{y}|^{-N-l-1})$ and $\psi(\mathbf{y}) = O(|\mathbf{y}|^l)$; therefore, the integrand is of order $O(|\mathbf{y}|^{-N-1})$ at infinity and $\phi_2(\mathbf{x})$ is defined for all $\mathbf{x} \in (D - \partial D)$.

We have to establish the rules for the first and second derivative of $\phi_2(\mathbf{x})$ for all $\mathbf{x} \in D$ and also the equality

$$-\nabla_{\mathbf{x}}^2 \int_{D_2} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}) = \psi_2(\mathbf{x}) \tag{20}$$

since $g(l+2, \mathbf{y}, -\mathbf{x})$ is singular and D_2 is unbounded. We can either follow the method in Kellogg's book [22, pp. 150–156] or apply Harnack's convergence theorem. The latter approach is shorter and is adopted here.

Choose an arbitrary point $\mathbf{q} \in (D - \partial D)$. As D_2 overlaps $D \cap S(\mathbf{0}, 3/2)$, there is an $\eta > 0$ such that at least $S(\mathbf{q}, \eta) \subset D_2$ or $S(\mathbf{q}, \eta) \subset (D \cap S(\mathbf{0}, 3/2))$. We then suppose that $\psi(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta))$.

We first suppose that $S(\mathbf{q}, \eta) \subset D_2$, hence $\psi_2(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta))$. Take an arbitrary point $\mathbf{w} \in S(\mathbf{q}, \eta)$ and choose a constant $\lambda > 0$ such that $S(\mathbf{w}, \lambda) \subset S(\mathbf{q}, \eta)$. The differentiability of $\phi_2(\mathbf{x})$ in $S(\mathbf{w}, \lambda)$ will be examined. Write $\phi_2(\mathbf{x})$ as the sum of two integrals, $\phi_{21}(\mathbf{x})$ and $\phi_{22}(\mathbf{x})$, of $\psi_2(\mathbf{x})$ over $S(\mathbf{w}, \lambda)$ and $D_2 - S(\mathbf{w}, \lambda)$, respectively. The first,

$$\phi_{21}(\mathbf{x}) \equiv -\int_{S(\mathbf{w},\lambda)} [h(\mathbf{y}-\mathbf{x}) - \sum_{k=0}^{l+2} (k!)^{-1} d^k h(\mathbf{y}, -\mathbf{x})]\psi_2(\mathbf{y})\, d\tau(\mathbf{y}),$$

is over a bounded region with a singular kernel caused by $|\mathbf{y}-\mathbf{x}|^{2-N}$ and is covered by Lemma 2 and the results of Lemma 1 regarding $\sum_{k=0}^{l+2}(k!)^{-1}d^k h(\mathbf{y}, -\mathbf{x})$. Hence, $\phi_{21}(\mathbf{x})$ satisfies Eqs. (16) to (19) with D replaced by $S(\mathbf{w}, \lambda)$ and $\{h(\mathbf{y}-\mathbf{x})[1 - u(|\mathbf{y}|-2)] + g(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}|-2)\}$ by $g(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}|-2)$. In particular, we have

$$\begin{aligned}
\nabla_{\mathbf{x}}^2 \phi_{21}(\mathbf{x}) &= -\nabla_{\mathbf{x}}^2 \int_{S(\mathbf{w},\lambda)} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}) \\
&= -\nabla_{\mathbf{x}}^2 \int_{S(\mathbf{w},\lambda)} h(\mathbf{y}-\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}) \\
&\quad - \nabla_{\mathbf{x}}^2 \int_{S(\mathbf{w},\lambda)} \left(\sum_{k=0}^{l+2} \frac{1}{k!} d^k h(\mathbf{y}, -\mathbf{x})\right) \psi_2(\mathbf{y})\, d\tau(\mathbf{y}) \\
&= \psi_2(\mathbf{x}) - \int_{S(\mathbf{w},\lambda)} \nabla_{\mathbf{x}}^2 \left(\sum_{k=0}^{l+2} \frac{1}{k!} d^k h(\mathbf{y}, -\mathbf{x})\right) \psi_2(\mathbf{y})\, d\tau(\mathbf{y}) = \psi_2(\mathbf{x}).
\end{aligned} \tag{21}$$

Differentiation has been brought under the fourth integral sign since the summation $\psi_2(\mathbf{y}) \sum_{k=0}^{l+2} (k!)^{-1} d^k h(\mathbf{y}, -\mathbf{x})$ is infinitely differentiable with respect to $\mathbf{x}$ and these derivatives are continuous for $(\mathbf{x}, \mathbf{y}) \in \overline{S}(\mathbf{w}, \lambda) \times \overline{S}(\mathbf{w}, \lambda)$ where the set is compact (Apostol [36, p. 167]). We also note that $\phi_{21}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{n+2}(S(\mathbf{w}, \lambda))$. The second integral $\phi_{22}(\mathbf{x})$ is over an unbounded region but with a bounded, infinitely differentiable kernel and is considered below.

Let $\{\rho_m\}$ be an unbounded increasing sequence with $\rho_1 > |\mathbf{q}| + \eta$. The function

$$\xi_m(\mathbf{x}) \equiv -\int_{D_2 \cap S(\mathbf{0}, \rho_m) - S(\mathbf{w}, \lambda)} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}) \tag{22}$$

is harmonic for $x \in \overline{S}(\mathbf{w}, \lambda/2)$ by Lemma 1. $\xi_m(\mathbf{x})$ can be differentiated under the integral sign since $g(l+2, \mathbf{y}, -\mathbf{x})$ is infinitely differentiable with respect to $\mathbf{x}$ and these derivatives are continuous for $(\mathbf{x}, \mathbf{y}) \in \overline{S}(\mathbf{w}, \lambda/2) \times ((D_2 \cup \partial D_2) \cap \overline{S}(\mathbf{0}, \rho_m) - S(\mathbf{w}, \lambda))$, where the set is compact.

Let $x \in \overline{S}(\mathbf{w}, \lambda/2)$ and let $m \to \infty$; then ρ_m also tends to infinity. Since $|g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})|$ is bounded for large $|\mathbf{y}|$ by $M|\mathbf{y}|^{-N-1}$ with some M independent of $(\mathbf{x}, \mathbf{y})$, the sequence of harmonic functions $\{\xi_m(\mathbf{x})\}$ converges uniformly on $\overline{S}(\mathbf{w}, \lambda/2)$ to the limit function $\phi_{22}(\mathbf{x}) \in C^\infty(\overline{S}(\mathbf{w}, \lambda/2))$ given by

$$\phi_{22}(\mathbf{x}) = -\int_{D_2 - S(\mathbf{w}, \lambda)} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}). \tag{23}$$

By Harnack's convergence theorem (Kellogg [22, p. 248]), $\xi_m(\mathbf{x})$ together with all of its derivatives converge on $\overline{S}(\mathbf{w}, \lambda/2)$ to $\phi_{22}(\mathbf{x})$ and its respective derivatives and $\phi_{22}(\mathbf{x})$ is harmonic in $\overline{S}(\mathbf{w}, \lambda/2)$. Hence

$$\begin{aligned}
-\nabla_{\mathbf{x}}\phi_{22}(\mathbf{x}) &\equiv -\nabla_{\mathbf{x}}\left(\lim_{m\to\infty}\xi_m(\mathbf{x})\right) = -\lim_{m\to\infty}(\nabla_{\mathbf{x}}\xi_m(\mathbf{x}))\\
&= \lim_{m\to\infty}\int_{D_2 \cap S(\mathbf{0}, \rho_m) - S(\mathbf{w}, \lambda)} \nabla_{\mathbf{x}} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y})\\
&= \int_{D_2 - S(\mathbf{w}, \lambda)} \nabla_{\mathbf{x}} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}),
\end{aligned} \tag{24}$$

$$\begin{aligned}
-\frac{\partial^2}{\partial x_i \partial x_j}\phi_{22}(\mathbf{x}) &\equiv -\frac{\partial^2}{\partial x_i \partial x_j}\left(\lim_{m\to\infty}\xi_m(\mathbf{x})\right) = -\lim_{m\to\infty}\left(\frac{\partial^2}{\partial x_i \partial x_j}\xi_m(\mathbf{x})\right)\\
&= \lim_{m\to\infty}\int_{D_2 \cap S(\mathbf{0}, \rho_m) - S(\mathbf{w}, \lambda)} \frac{\partial^2}{\partial x_i \partial x_j} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y})\\
&= \int_{D_2 - S(\mathbf{w}, \lambda)} \frac{\partial^2}{\partial x_i \partial x_j} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y}),
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
-\nabla^2\phi_{22}(\mathbf{x}) &= \lim_{m\to\infty}\int_{D_2 \cap S(\mathbf{0}, \rho_m) - S(\mathbf{w}, \lambda)} \nabla^2_{\mathbf{x}} g(l+2, \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})\, d\tau(\mathbf{y})\\
&= \lim_{m\to\infty} 0 = 0,
\end{aligned} \tag{26}$$

by Eq. (9) and Lemmas 1 and 2. Differentiation has been brought under the integral sign for the third members of Eqs. (24) and (25) since $g(l+2), \mathbf{y}, -\mathbf{x})\psi_2(\mathbf{y})$ is infinitely differentiable with respect to $\mathbf{x}$ and these derivatives are continuous for $(\mathbf{x}, \mathbf{y}) \in \overline{S}(\mathbf{w}, \lambda/2) \times ((D_2 \cup \partial D_2) \cap \overline{S}(0, \rho_m) - S(\mathbf{w}, \lambda))$ where the set is compact. Since $\mathbf{w}$ is an arbitrary point of $S(\mathbf{q}, \eta)$, it follows that $\phi_{22}(\mathbf{x})$ is harmonic in

$S(\mathbf{q}, \eta)$. Consequently, $\phi_{22}(\mathbf{x}) \in C^{\infty}(S(\mathbf{q}, \eta))$ and the differentiability of $\phi_{21}(\mathbf{x})$ determines that of $\phi_2(\mathbf{x})$. Finally, since $\phi_{22}(\mathbf{x})$ is harmonic, Eq. (21) gives

$$\nabla_{\mathbf{x}}^2\phi_2(\mathbf{x}) = \nabla_{\mathbf{x}}^2\phi_{21}(\mathbf{x}) + \nabla_{\mathbf{x}}^2\phi_{22}(\mathbf{x}) = \nabla_{\mathbf{x}}^2\phi_{21}(\mathbf{x}) = \psi_2(\mathbf{x}). \tag{27}$$

Hence $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) + \phi_{21}(\mathbf{x}) + \phi_{22}(\mathbf{x})$ satisfies Eqs. (16), (17), (18), and (19).

Secondly, we can have a similar but simpler argument when $S(\mathbf{q}, \eta) \subset (D \cap S(\mathbf{0}, 3/2))$ since $\phi_{21}(\mathbf{x}) \equiv 0$ in this case. The theorem has been proved. (It could also be generalized to state that if $\psi(\mathbf{x}) \in C^0(D \cup \partial D)$ then $\phi(\mathbf{x}) \in C^1(D \cup \partial D)$ with only some simple change of the proof.)

4. N-dimensional Helmholtz's decomposition theorem.

LEMMA 3. Let $\mathbf{a}(\mathbf{x})$ be a twice differentiable vector function of $\mathbf{x}$. Then the $(N-2)$th order antisymmetric tensor $\mathbf{b}(\mathbf{x})$ defined by

$$b_{[\hat{i}\hat{j}]} = (-1)^{i+j}\left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) \tag{28}$$

satisfies

$$\nabla^2 a_j - \frac{\partial}{\partial x_j}\nabla \cdot \mathbf{a} = (-1)^{j-1}(d\mathbf{b})_{12\cdots\hat{j}\cdots N}. \tag{29}$$

This is the N-dimensional generalization of $\nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a} = \nabla \times (\nabla \times \mathbf{a})$.

Proof. The $(N-2)$-form

$$\mathbf{b}(\mathbf{x}) = \sum_{[\hat{i}\hat{j}] \in K^{N-2}} b_{12\cdots\hat{i}\cdots\hat{j}\cdots N}(\mathbf{x})\, d\mathbf{x}_{[\hat{i}\hat{j}]} \tag{30}$$

has for its exterior derivative $d\mathbf{b}$ the $(N-1)$-form

$$d\mathbf{b} = \sum_{[\hat{i}\hat{j}] \in K^{N-2}} (-1)^{i-1}\frac{\partial b_{12\cdots\hat{i}\cdots\hat{j}\cdots N}}{\partial x_i}\, d\mathbf{x}_{[\hat{j}]} + \sum_{[\hat{i}\hat{j}] \in K^{N-2}} (-1)^{j-2}\frac{\partial b_{12\cdots\hat{i}\cdots\hat{j}\cdots N}}{\partial x_j}\, d\mathbf{x}_{[\hat{i}]}.$$

Define

$$B_{ij} \equiv -B_{ji} \equiv b_{12\cdots\hat{i}\cdots\hat{j}\cdots N} \quad \text{for } i < j, \qquad B_{ij} = 0 \quad \text{for } i = j, \tag{31}$$

and note that $B_{ij} = (-1)^{i+j}(\partial a_j/\partial x_i - \partial a_i/\partial x_j)$ for all $1 \le i, j \le N$. We have

$$d\mathbf{b} = \sum_{i=1}^{N}\sum_{j=1}^{N}(-1)^{i-1}\frac{\partial B_{ij}}{\partial x_i}\, d\mathbf{x}_{[\hat{j}]} = \sum_{i=1}^{N}\sum_{j=1}^{N}(-1)^{i-1}(-1)^{i+j}\frac{\partial}{\partial x_i}\left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}\right) d\mathbf{x}_{[\hat{j}]}, \tag{32}$$

which gives Eq. (29).

THEOREM 2. Let $D \subset R^N$ be an infinite regular region and let $\mathbf{f}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{0,\alpha}(D - \partial D)$ be a vector function bounded at infinity by $O(|\mathbf{x}|^l)$, l a constant. Then $\mathbf{f}(\mathbf{x})$ can be written as

$$\mathbf{f}(\mathbf{x}) = \nabla\theta(\mathbf{x}) + \sum_{j=1}^{N}\mathbf{e}_j(-1)^{j-1}(d\mathbf{b})_{12\cdots\hat{j}\cdots N} \tag{33}$$

in $D - \partial D$ for some functions $\theta(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^0(D \cup \partial D)$. For every $S(\mathbf{q}, \eta) \subset (D - \partial D)$ and any integer $n \geq 0$, the scalar and the $(N-2)$th-order tensorial functions $\theta(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ satisfy

$$\theta(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)) \quad \text{if } \mathbf{f}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)). \tag{34}$$

Proof. Theorem 1 gives a solution $\mathbf{a}(\mathbf{x}) \in C^1(D \cup \partial D)$ to $\nabla^2 \mathbf{a} = \mathbf{f}$. For every $S(\mathbf{q}, \eta) \subset (D - \partial D)$ and any $n \geq 0$, $\mathbf{a}(\mathbf{x})$ satisfies

$$\mathbf{a}(\mathbf{x}) \in C^{n+2}(S(\mathbf{q}, \eta)) \quad \text{if } \mathbf{f}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)). \tag{35}$$

Equation (28) defines an $(N-2)$ th-order tensor $\mathbf{b}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{n+1} S(\mathbf{q}, \eta)$. Lemma 3 then gives Eq. (29). Define $\theta(\mathbf{x}) \equiv \nabla_{\mathbf{x}} \cdot \mathbf{a}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{n+1} S(\mathbf{q}, \eta)$ and recall that $\nabla^2 \mathbf{a} = \mathbf{f}$; we have the required Eq. (33). For every $S(\mathbf{q}, \eta) \subset (D - \partial D)$ and any $n \geq 0$, we then use Eq. (35) to obtain

$$\theta(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)) \quad \text{if } \mathbf{f}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)).$$

$\nabla \theta(\mathbf{x})$ is curl-free. The divergence of the vector

$$s(\mathbf{x}) \equiv \sum_{j=1}^{N} \mathbf{e}_j (-1)^{j-1} (d\mathbf{b})_{12 \cdots \hat{j} \cdots N}$$

is calculated from

$$(\nabla \cdot \mathbf{s})\, d\mathbf{x}_{12 \cdots N} = \sum_{j=1}^{N} \{\partial[(d\mathbf{b})_{12 \cdots \hat{j} \cdots N}]/(\partial x_j)\} (-1)^{j-1}\, dx_j \mathbf{e}_j \wedge d\mathbf{x}_{[\hat{j}]} = d(d\mathbf{b}).$$

The last member is identically zero by direct calculation (Poincaré's Lemma; see, e.g., Flanders [37]). The theorem has been proved.

COROLLARY 1. Let $D \subset R^3$ be an infinite regular region and let $\mathbf{f}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{0,\alpha}(D - \partial D)$ be a vector function bounded at infinity by $O(|x|^l)$, l a constant. Then $\mathbf{f}(\mathbf{x})$ can be written as

$$\mathbf{f}(\mathbf{x}) = \nabla \theta(\mathbf{x}) + \sum_{j=1}^{3} \mathbf{e}_j (-1)^{j-1} (d\mathbf{b})_{12 \cdots \hat{j} \cdots 3} = \nabla \theta(\mathbf{x}) + \nabla \times \mathbf{b}(\mathbf{x}) \tag{36}$$

in $D - \partial D$ for some functions $\theta(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^0(D \cup \partial D)$. For every $S(\mathbf{q}, \eta) \subset (D - \partial D)$ and any integer $n \geq 0$, the scalar and the vectorial functions $\theta(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ satisfy

$$\theta(\mathbf{x}), \mathbf{b}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)) \quad \text{if } \mathbf{f}(\mathbf{x}) \in C^{n+1}(S(\mathbf{q}, \eta)). \tag{37}$$

5. Applications.

1. The homogeneous Lamé equation is

$$(1 - 2\nu)\nabla^2 \mathbf{u}(\mathbf{x}) + \nabla(\nabla \cdot \mathbf{u}) = \mathbf{0} \tag{38}$$

where ν is a constant (Poisson's ratio). Mindlin [38] applied Helmholtz's decomposition (twice) to $\mathbf{u}(\mathbf{x})$ to show that the general (Papkovich-Neuber) solution to Eq. (38) is

$$\mathbf{u}(\mathbf{x}) = 4(1 - \nu)\mathbf{b} - \nabla(\mathbf{x} \cdot \mathbf{b} + \phi) \quad \text{where } \nabla^2 \mathbf{b}(\mathbf{x}) = \mathbf{0} \text{ and } \nabla^2 \phi(\mathbf{x}) = 0. \tag{39}$$

Theorem 2 shows that the solution is applicable whenever $|\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^l)$, l a constant, at infinity.

2. The solvability of the "Legendre equation", found in the studies of Legendre functions,

$$\frac{1}{\cos^2\gamma}\left(\frac{\partial^2}{\partial\theta^2} + \cos\gamma\frac{\partial}{\partial\gamma}\left(\cos\gamma\frac{\partial}{\partial\gamma}\right)\right)\chi + \lambda\chi(\theta, \gamma) = b(\theta, \gamma), \tag{40}$$

where $b(\theta, \gamma)$ is a given function, is a central device in the proof of the generality of the above Papkovich-Neuber solution in terms of only $\mathbf{b}(\mathbf{x})$ (i.e., with $\phi(\mathbf{x})$ omitted; see Tran-Cong [39]). Its solution is complicated by any other method but is fairly simple using the two-dimensional analogue of Theorem 1: By setting $\eta = \ln|[1+\tan(\gamma/2)]/[1-\tan(\gamma/2)]|$, $a(\theta, \eta) = b(\theta, \gamma)$, it becomes

$$\left(\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\eta^2}\right)\xi(\theta, \eta) + \lambda\frac{4e^{2\eta}}{(e^{2\eta}+1)^2}\xi(\theta, \eta) = \frac{4e^{2\eta}}{(e^{2\eta}+1)^2}a(\theta, \eta). \tag{41}$$

Define $\mathbf{x} \equiv (x_1, x_2) = (\theta, \eta)$, $h_2(\mathbf{x}) \equiv (4\pi)^{-1}\ln(x_1^2 + x_2^2)$, $g_2(l+2, \mathbf{y}, -\mathbf{x}) \equiv h_2(\mathbf{x}-\mathbf{y}) - \sum_{k=0}^{l+2}(k!)^{-1}d^k h_2(\mathbf{y}, -\mathbf{x})$. The analogue of Eq. (15),

$$\xi(\mathbf{x}) = -\iint_S \{h_2(\mathbf{x}-\mathbf{y})[1-u(|\mathbf{y}|-2)] + g_2(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}|-2)\}\zeta(\mathbf{y})\,dy_1\,dy_2, \tag{42}$$

turns it into a Fredholm integral equation with a weak singularity

$$\zeta(x_1, x_2) + \lambda\iint K((x_1, x_2), (y_1, y_2))\zeta(y_1, y_2)\,dy_1\,dy_2 = F(x_1, x_2) \tag{43}$$

where $K(\mathbf{x}, \mathbf{y}) = [8\pi e^{2x_2}/(e^{2x_2}+1)^2]\{h_2(\mathbf{x}-\mathbf{y})[1-u(|\mathbf{y}|-2)] + g_2(l+2, \mathbf{y}, -\mathbf{x})u(|\mathbf{y}|-2)\}$ and $F(x_1, x_2) = [4e^{2x_2}/(e^{2x_2}+1)^2]a(x_1, x_2)$. The solution $\zeta(x_1, x_2)$ to Eq. (43) is periodic in $x_1 \equiv \theta$, and Eq. (42) specifies the correspondence between $\zeta(\mathbf{x})$ and $\xi(\mathbf{x})$.

3. Consider next another interesting application of Theorem 1 and the technique of Theorem 2.

Theorem 3. Let the infinite regular region $D \subset R^N$ be star-shaped, which has the property that every point in D can be joined to the origin by a straight line segment that lies totally in D, and let $\psi(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^\infty(D - \partial D)$ be a harmonic function in $D \cup \partial D$ and bounded at infinity by $O(|\mathbf{x}|^l)$, l a constant. Then $\psi(\mathbf{x})$ can be expressed as

$$\psi = \nabla \cdot \mathbf{a} \quad \text{where } \nabla^2\mathbf{a} = \mathbf{0} \text{ and } \mathbf{a}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^\infty(D - \partial D). \tag{44}$$

Proof. Consider the $(N-1)$-form $\mathbf{c} = \sum_{[\hat{j}]\in K^{N-1}}(-1)^{j-1}(\partial\psi/\partial x_j)\,dx_{[\hat{j}]}$. Since $\nabla^2\psi = 0$, we have

$$0 = [\nabla\cdot(\nabla\psi)]\,d\mathbf{x}_{12\cdots N} = \sum_{j=1}^{N}\frac{\partial}{\partial x_j}[(\mathbf{c})_{12\cdots\hat{j}\cdots N}](-1)^{j-1}\,dx_j\mathbf{e}_j \wedge d\mathbf{x}_{[\hat{j}]} \equiv d\mathbf{c}.$$

By the Converse of Poincaré's Lemma (see Flanders [37]), $\mathbf{c}$ is exact, i.e., $\mathbf{c} = d\mathbf{b}$ where $\mathbf{b}(\mathbf{x}) \in C^{\infty}(D - \partial D)$ is an $(N-2)$-form, and we can write

$$\nabla\psi = \mathbf{e}_j(-1)^{j-1}(d\mathbf{b})_{12\cdots\hat{j}\cdots N}. \tag{45}$$

Define a scalar function $\phi(\mathbf{x}) \in C^1(D \cup \partial D) \cap C^{\infty}(D - \partial D)$ and an $(N-2)$-form $\mathbf{s}(\mathbf{x}) \in C^1(D \cup \partial D) \cap C^{\infty}(D - \partial D)$ by

$$\phi(\mathbf{x}) \equiv -\int_D \{h(\mathbf{y}-\mathbf{x})[1-u(|\mathbf{y}|-2)] + g(l+2,\, \mathbf{y},\, -\mathbf{x})u(|\mathbf{y}|-2)\}\psi(\mathbf{y})\, d\tau(\mathbf{y}),$$

$$\mathbf{s}(\mathbf{x}) \equiv -\int_D \{h(\mathbf{y}-\mathbf{x})[1-u(|\mathbf{y}|-2)] + g(l+2,\, \mathbf{y},\, -\mathbf{x})u(|\mathbf{y}|-2)\}\mathbf{b}(\mathbf{y})\, d\tau(\mathbf{y}).$$

Then the function $\mathbf{a}(\mathbf{x}) \equiv \nabla\phi - \mathbf{e}_j(-1)^{j-1}(d\mathbf{s})_{12\cdots\hat{j}\cdots N}$ satisfies

$$\nabla\cdot\mathbf{a} = \nabla^2\phi - (d(d\mathbf{s}))_{12\cdots N} = \nabla^2\phi = \psi.$$

We also have

$$\nabla^2\mathbf{a} = \nabla(\nabla^2\phi) - \mathbf{e}_j(-1)^{j-1}(d(\nabla^2\mathbf{s}))_{12\cdots\hat{j}\cdots N} = 0,$$

since $\nabla^2\phi = \psi$ and $\nabla^2\mathbf{s} = \mathbf{b}$. Therefore $\mathbf{a}(\mathbf{x}) \in C^0(D \cup \partial D) \cap C^{\infty}(D - \partial D)$. The theorem has been proved. It can even be generalized so that D needs only be deformable to star shape.

When $N = 3$, the theorem states that every harmonic function $\psi(\mathbf{x})$ in a region $D \subset R^3$ with m internal surfaces $S_1, S_2, \ldots, S_m$ is expressible as

$$\psi(\mathbf{x}) = \sum_{k=1}^{m} \frac{q_k}{|\mathbf{x}-\mathbf{c}_k|} + \nabla\cdot\mathbf{a} \quad \text{where } \nabla^2\mathbf{a} = \mathbf{0}, \tag{46}$$

with $\mathbf{c}_k$ and q_k, $1 \le k \le m$, being, respectively, the constant position vector and the associated scalar constant of an interior point of the corresponding internal surface S_k. D need not be star-shaped for $N = 3$ since Stevenson's proof [40], instead of the Converse of Poincaré's Lemma, is used to write $\nabla\psi = \nabla\times\mathbf{b} + \sum_{k=1}^{m} q_k(\mathbf{c}_k - \mathbf{x})/|\mathbf{c}_k - \mathbf{x}|^2$. The gradient of Eq. (46) also gives the first part of Theorem IV of Weyl's paper: The gradient of every harmonic function $\psi(\mathbf{x})$ is equal to $\nabla\times\mathbf{b} + \sum_{k=1}^{m} q_k(\mathbf{c}_k - \mathbf{x})/|\mathbf{c}_k - \mathbf{x}|^2$ for some $\mathbf{b}(x)$.

References

[1] H. Helmholtz, *Ueber Integral der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen*, Crelle **55**, 25–55 (1858)

[2] G. G. Stokes, *On the dynamical theory of diffraction*, Cambridge Philos. Trans. **9**, 1–62 (1849)

[3] A. Sommerfeld, *Mechanics of Deformable Bodies*, Academic Press, New York, 1950

[4] H. Lass, *Vector and Tensor Analysis*, McGraw-Hill, Kogakusha, Tokyo, 1950

[5] R. Aris, *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice-Hall, Englewood Cliffs, NJ, 1962

[6] R. M. Bowen and C. C. Wang, *Introduction to vectors and tensors*, Vectors and Tensor Analysis, Vol. 2, Plenum Press, New York, 1976

[7] H. B. Phillips, *Vector Analysis*, John Wiley & Sons, New York, 1933

[8] C. E. Weatherburn, *Advanced Vector Analysis*, G. Bell & Sons, London, 1924

[9] O. Blumenthal, *Uber die Zerlegung unendlicher Vectorfelder*, Math. Ann. **61**, 235–250 (1905)

[10] M. E. Gurtin, *On Helmholtz's Theorem and the completeness of the Papkovich-Neuber stress functions for infinite domains*, Arch. Rational Mech. Anal. **9**, 225–233 (1962)
[11] O. Nikodym, *Sur un théorème de M. S. Zaremba concernant les fonctions harmoniques*, J. de Math. **12**, 95–108 (1933)
[12] K. Friedrichs, *On differential operators in Hilbert spaces*, Amer. J. Math. **61**, 523–544 (1939)
[13] H. Weyl, *The method of orthogonal projection in potential theory*, Duke Math. J. **7**, 411–444 (1940)
[14] E. B. Bykhovsky and N. V. Smirnov, *On orthogonal expression of the space of vector functions which are square-summable over a given domain*, Trudy Mat. Inst. Steklova **59**, 5–36 (1960)
[15] D(aisuke) Fujiwara and H(iroko) Morimoto, *An L_r-theorem of the Helmholtz decomposition of vector fields*, J. Fac. Sci. Univ. Tokyo Sect. I Math. **24**, 685–700 (1977)
[16] M. E. Gurtin, *The linear theory of elasticity*, Mechanics of Solids, vol. 2, edited by C. Truesdell, Springer-Verlag, Berlin, 1973
[17] R. F. Millar, *On the completeness of the Papkovich potentials*, Quart. Appl. Math. **41**, 385–393 (1984)
[18] G. J. Hirasaki, *Boundary conditions on the vector and scalar potentials in viscous three-dimensional hydrodynamics*, Quart. Appl. Math. **28**, 293–296 (1970)
[19] Y. A. S. Aregrebesola and D. M. Burley, *The vector and scalar potential method for the numerical solution of two- and three-dimensional Navier-Stokes equations*, J. Comp. Phys. **24**, 398–415 (1977)
[20] S. M. Richardson and A. R. H. Cornish, *Solution of three-dimensional incompressible flow problems*, J. Fluid Mech. **82**, 309–319 (1977)
[21] L. Morino, *Helmholtz decomposition revisited: Vorticity generation and trailing edge condition*, Comp. Mech. **1**, 65–90 (1986)
[22] O. D. Kellogg, *Foundations of Potential Theory*, Springer, New York, 1929; Dover, New York, 1953
[23] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, 1962
[24] M. Schechter and B. Simon, *Unique continuation for Schrödinger operators with unbounded potentials*, J. Math. Anal. Appl. **77**, 482–492 (1980)
[25] W. O. Amrein, A. M. Berthier, and V. Georcescu, *L^p-inequalities for the Laplacian and unique continuation*, Ann. Inst. Fourier (Grenoble) **31**, 153–168 (1981)
[26] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. (2) **121**, 463–494 (1985)
[27] L. Hormander, *The Analysis of Linear Partial Differential Operators* II, Springer-Verlag, Berlin, 1983
[28] R. L. Fosdick, *On the vector potential and the representation of a polyharmonic function in n-dimensions*, J. Math. Mech. **14**, 573–587 (1965)
[29] T(on) Tran-Cong, *Notes on some relationship between the Galerkin, Papkovich-Neuber and Naghdi-Hsu solutions in linear elasticity*, Mech. Res. Comm. **8**, 207–211 (1981)
[30] T(on) Tran-Cong and J. R. Blake, *General solutions of the Stokes's flow equations*, J. Math. Anal. Appl. **90**, 72–84 (1982)
[31] E. Schmidt, *Bemerkung zur Potentialtheorie. Mathematische Abhandlugen Hermann Amandus Schwarz zu seinem funfzigjahrigen Doktorjubilaum*, Springer, Berlin, 1914
[32] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order, 2nd ed.* Springer-Verlag, Berlin, 1983
[33] S. G. Mihlin, *Singular integral equations*, Translation No. 24, Amer. Math. Soc., Providence, RI, 1950
[34] A. Calderon and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88**, 85–139 (1952)
[35] M. H. Taibleson, *The preservation of Lipschitz spaces under singular integral operators*, Studia Math. **24**, 107–111 (1963)
[36] T. Apostol, *Mathematical Analysis, 2nd. ed.*, Addison-Wesley, Reading, MA, 1974
[37] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Academic Press, New York, 1963
[38] R. D. Mindlin, *Note on the Galerkin and Papkovich stress functions*, Bull. Amer. Math. Soc. **42**, 373–376 (1936)
[39] T(on) Tran-Cong, *On the completeness of the Papkovich-Neuber solution*, Quart. Appl. Math. **47**, 645–659 (1989)
[40] A. F. Stevenson, *Note on the existence and determination of a vector potential*, Quart. Appl. Math. **12**, 194–198 (1954)