THE NEUTRINO THEORY OF LIGHT.

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AND

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Received March 18, 1936.

1. *Introduction.*

THE most important contribution during the last years to the fundamental conceptions of theoretical physics seems to be the development of a new theory of light which contains the acknowledged one as a limiting case, but connects the optical phenomena with those of a very different kind, radioactivity. The idea that the photon is not an elementary particle but a secondary one, composed of simpler particles, has been first mentioned by P. Jordan[1]. His argument was a statistical one, based on the fact, that photons satisfy the Bose-Einstein statistics. It is known from the theory of composed particles as nuclei, atoms or molecules, that this statistics may appear for such systems, which are compounds of elementary particles satisfying the Fermi-Dirac statistics (*e.g.*, electrons or protons). After the discovery of the neutrino, de Broglie[2] suggested that a photon $h\nu$ is composed of a "neutrino" and an "anti-neutrino," each having the energy $\frac{1}{2}h\nu$. He has developed some interesting mathematical relations between the wave equation of a neutrino (which he assumed to be Dirac's equation for a vanishingly small rest-mass) and Maxwell's field equations. But de Broglie has not touched the central problem, namely, as to how the Bose-Einstein statistics of the photons arises from the Fermi-statistics of the neutrinos. This question cannot be solved in the same way as in the cases mentioned above (material particles) where it is a consequence of considering the composed system as a whole neglecting internal motions. This is possible because of the strong forces keeping the primary particles together. But in the case of neutrinos we have no knowledge of such forces, and it would contradict the simplicity of de Broglie's idea to introduce them.

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[1] P. Jordan, "Die Lichtquanten hypothese," *Erg. der Exakt. Naturw.*, 1928, **17**, 158. P. Jordan, *Zeits. f. Phys.*, 1935, **93**, 464; 1936, **98**, 709 and 759; 1936, **99**, 109.

[2] L. de Broglie, *C.R.*, 1932, **195**, 536, 577, 862; 1933, **197**, 1377; 1934, **198**, 135; 1934, **199**, 445, 1165. A coherent representation is to be found in his book *Une Nouvelle Conception de la Lumiere* (181, Act. Sci. et Ind.). Further L. de Broglie and J. Winter, *C.R.*, 1934, **199**, 813.

This fundamental problem has been solved by Jordan. He has shown that no forces between neutrinos are required but the way of their interaction with electric charges leads to the effect that pairs of neutrinos behave generally as photons. Jordan has formulated the principles involved and has proved rather deep mathematical theorems expressing the relation of neutrinos and photons. The fuller development of the mathematical method is due to R. de L. Kronig[3] who is working in close co-operation with Jordan. We shall give here a report on the present situation, as far as it has come to our knowledge and hope that this will be welcome to the readers, not only because the original papers are scattered over several periodicals but are extremely difficult to read as they contain some apparent contradictions which have not been avoided even in the latest paper.

We disagree with Jordan and Kronig only in one essential point. It is that we see no reason to introduce the spin of the neutrino, but that we can describe the difference between the two kinds of neutrinos in the same way as the difference between electrons and positrons in Dirac's theory of holes.

We have tried to make this report comprehensive without much recourse to literature.

2. *The Neutrino.*

The particles which physics considers to-day as elementary can be ordered corresponding to their masses in 3 groups, each containing two types :

1. Particles of great mass : proton, neutron ;
2. Particles of small mass : electron, positron ;
3. Particles of zero mass : photon, neutrino.

The central problem of future physics is the explanation of the existence of these different types of particles and the derivation of their properties from a fundamental principle. As to the first two groups, we think that the solution must lie in the direction of a non-linear field theory, as has been tried to develop.[4] But the particles of the third group correspond classically to the case of very weak field, where the non-linearity plays no rôle. This limiting case was generally assumed to be known with any desired accuracy, as represented by the quantised field equations of Maxwell. But the discovery of the neutron has shaken this conviction and we are compelled to revise the very principles even of the limiting case, before we can hope to tackle the (probably non-linear) laws governing the higher particles.

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[3] R. de L. Kronig, *Physica,* 1935, 2, 491, 854, 968.

[4] M. Born, *Proc. Roy. Soc.,* (A), 1934, 143, 410.

The experimental facts which lead Pauli to the assumption of neutrinos are connected with the β-decay of radio-active nucleii. The spectrum of the emitted electrons (or positrons in the case of induced radio-activity) is continuous and has a wide range of velocity. But the emitting nucleus and the resulting nucleus are in every respect definite systems. All attempts to account for this fact by a simultaneous emission of γ-rays have failed.

There are only two ways of explanation : either the law of the conservation of energy does not hold in these cases or there are particles emitted which cannot be detected by our experimental methods, because of their extreme penetrating power.

To avoid the first unpleasant alternative,* Pauli has put forward the idea that very light, uncharged particles are emitted simultaneously with the electron when a neutron is converted into a proton (or *vice versa*), and he has called them *neutrinos*. It is not quite hopeless to prove their existence by using very light atoms made radio-active with the help of Fermi's method of neutron bombardment. One could observe the recoil of these atoms when emitting the electrons and measure the energy and momentum of both the particles. If there is a third particle involved in the process, its energy E and momentum p should be connected by the relativistic relation $\frac{E^2}{c^2} - p^2 = m_0{}^2c^2$ where c is the velocity of light and m_0 its rest-mass. If m_0 turns out to be constant (for instance zero), there could be no doubt about the existence of the neutrino. Experiments of this kind have been undertaken in the Cavendish laboratory at Cambridge.[5]

Fermi[6] has shown that the hypothesis of neutrino emission for β-decay together with some simple and natural assumption about the interaction energy leads to a definite law of distribution for the emitted β-rays which is in good agreement with observation, if the rest-mass of the neutrino is taken as very small, very much smaller than that of the electron. It seems very probable that the rest-mass is exactly zero just as that of the photon.

These meagre facts are the experimental evidence on which the neutrino theory of light is based. *The leading idea is that the neutrinos are the primary*

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* New experimental results of R. S. Shankland (*Phy. Rev.*, 1936, **49**, 8) on the Compton Effect (simultaneous emission of photons and electrons) indicate a failure of the conservation laws. P. A. M. Dirac has, therefore, concluded (*Nature*, 1936, **137**, 298) that the said laws have to be given up. Nevertheless, we think it worth while to study carefully the possible way out of this dilemma pointed out by the neutrino theory.

[5] Bainbridge [*Science*, 1936, **83**, 38 (Suppl.)] claims to have a direct proof of the existence of the neutrino from considerations about the stability of isotopes.

[6] E. Fermi, *Nuovo Cimento*, 1934, **11**, 1.

particles moving with the velocity of light, and that the processes usually described in terms of photons or light waves are really simultaneous actions of several neutrinos.

Jordan and de Kronig have followed de Broglie in the assumption that there are two kinds of neutrinos, neutrinos and anti-neutrinos, differing by their spin. This is the only point where we do not agree with them. *We have found that the whole theory could be developed by assuming only one kind of neutrino which can have positive or negative energy.* Since in the thermal equilibrium at absolute zero all negative states are occupied, the total negative energy becomes infinite. This inconvenience is overcome by using the number of unoccupied states as variables and to call them anti-neutrinos, just as in Dirac's theory of holes representing positrons. But whereas in the case of the electrons, the formulation of this theory of holes leads to great difficulties,[7] arising from the external fields which have to be taken into account, it is very simple and satisfactory here in the case of the neutrinos which are not attacked by external forces.

3. *Fermi Statistics of Neutrinos.*

We describe the motion of the neutrinos and anti-neutrinos with the help of two infinite sets of non-commuting variables α_κ, γ_κ which we numerate (for sake of convenience) with the help of half numbers, $\kappa = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\cdots$ We define the α_κ, γ_κ also for negative indices with the help of the relations:*

$$(1) \quad \alpha_{-\kappa} = \gamma_\kappa^{+}\,; \qquad \gamma_{-\kappa} = \alpha_\kappa^{+}, \qquad \kappa > 0$$

where the $^{+}$ means the adjoint operator.

The meaning of these variables can be described, using the correspondence principle, as Fourier-coefficients of wave functions; for instance in one dimension

$$(2) \quad \begin{cases} \psi\,(t - x/c) = \sum\limits_{\kappa=-\infty}^{\infty} \alpha_\kappa\, e^{2\pi i \nu_1 \kappa (t - x/c)}, \\ \chi\,(t - x/c) = \sum\limits_{\kappa=-\infty}^{\infty} \gamma_\kappa\, e^{2\pi i \nu_1 \kappa (t - x/c)} = \psi^{+}, \end{cases}$$

where ν_1 corresponds to the fundamental frequency of the waves in the finite space considered (Hohlraum). We can interpret κ as the positive or negative energy of the corresponding state in units of $h\nu_1$.

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[7] W. Heisenberg, *Zeits. f. Phy.*, 1934, **90**, 209; and 1936, **98**, 714.
P. A. M. Dirac, *Proc. Camb. Phil. Soc.*, 1934, **45**, 245; and *Quantum Mechanics*, 1935, Chap. 13.

* There is really no use of using two kinds of variables; we introduce the γ_κ only to be in conformity with Jordan and to have symmetry between neutrinos and anti-neutrinos.

The functions ψ, χ are not self-adjoint; but $\chi = \psi^{\dagger}$. Instead of using the states of negative energy, one can introduce the holes amidst the states; the wave function describing them is $\chi = \psi^{\dagger}$ and to each positive energy κ belongs a neutrino (ψ, or a_{κ}) and a hole or anti-neutrino (χ or γ_{κ}).

We postulate the following commutation laws:

$$(3)\quad \begin{cases} a_{\kappa}\, a_{\mu} + a_{\mu}\, a_{\kappa} = 0 \\ \gamma_{\kappa}\, \gamma_{\mu} + \gamma_{\mu}\, \gamma_{\kappa} = 0 \\ \gamma_{\kappa}\, a_{\mu} + a_{\mu}\, \gamma_{\kappa} = \delta_{\mu, -\kappa}. \end{cases}$$

The number of neutrinos and anti-neutrinos of energy κ is defined only for $\kappa > 0$ by the operators

$$(4)\quad \begin{cases} N_{\kappa}^{(+)} = a_{\kappa}^{+}\, a_{\kappa} = 1 - a_{\kappa}\, a_{\kappa}^{+} \\ N_{\kappa}^{(-)} = \gamma_{\kappa}^{+}\, \gamma_{\kappa} = 1 - \gamma_{\kappa}\, \gamma_{\kappa}^{+}. \end{cases}$$

There are two other expressions for each of these quantities with the help of (1).

It is well known that these operators (4) have only the eigenvalues 0, 1, which mean unoccupied or occupied state. As we do not suppose extended knowledge of literature, we shall explain this a little closer in the next section.

4. *Matrix Representation of the Fundamental Operators.*

We start with a simple case and define the matrices*

$$(5)\quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\quad a^{+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};\ \text{and } s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

which satisfy the following relations:

$$(6)\quad \begin{cases} a^2 = 0 & a^{+2} = 0 & s^2 = 1 \\ a^{+}a + aa^{+} = 1 & as + sa = 0 & a^{+}s + sa^{+} = 0. \end{cases}$$

The matrix a is useful to describe a system with one state which could be occupied by a particle or not. We define the "number of particles" in this state which is of course either 0 or 1, with the help of the matrix

$$(7)\quad a^{+}a = n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which is diagonal and has the eigenvalues 0 and 1

Thus from (5) it follows that

$$(8)\quad aa^{+} = 1 - n.$$

We remark that one has

$$(9)\quad s = 1 - 2n$$

but we do not enter in a more elaborate discussion of the interesting formalism connected with these matrices. Now we extend this method to systems with

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* They are closely connected with Pauli's spin matrices:

$$\sigma_x = a^{\dagger} + a;\ \sigma_y = i\,(a^{\dagger} - a);\ \text{and } \sigma_z = s.$$

more than one state, occupied or not, by introducing the notion of the direct product of matrices.

5. *Direct Product of Matrices.*

If $a = (a_{kl})$ and $b = (b_{mn})$ are two matrices, the direct product is given by

(10) $a \times b = (a_{kl} \cdot b_{mn})$

i.e., the elements are all products of any element of a with any element of b.

(11) $(a \times b)_{km,\, ln} = a_{kl} b_{mn}.$

If $a \times b = 0$ it follows that either $a = 0$ or $b = 0$.

Further

(12) $(a \times b)^+ = \{(a \times b)^+_{ln,\, km}\} = (a_{lk}{}^+ \, b_{nm}{}^+) = a^+ \times b^+.$

6. *Representation of α_K and γ_K.*

We now give a representation of the matrices α_K and γ_K fulfilling (3) with the help of the direct products of matrices consisting of the fundamental matrices a and s. This representation was discovered by Jordan and Wigner[8] who have also shown that the solution is unique apart from unitary transformations. Their representation is given by the scheme (next page).

Using the definition (10) of the direct product of matrices and the equations (6), (11) and (12), it can be easily verified that the representations of the fundamental operators given in (13) satisfy the commutation rules (3).

7. *A New Set of Operators a_K and c_K.*

We will now define, as Jordan has done, a new set of operators as functions of our fundamental operators α_K and γ_K.

They are

$$(15) \qquad \begin{aligned} a_K &= \frac{\alpha_K + \gamma_K}{\sqrt{2}}, \\ c_K &= \frac{\alpha_K - \gamma_K}{i\sqrt{2}}. \end{aligned}$$

From (15) we can express α_K and γ_K as functions of a_K and c_K :

$$(16) \qquad \begin{aligned} \alpha_K &= \frac{a_K + ic_K}{\sqrt{2}}, \\ \gamma_K &= \frac{a_K - ic_K}{\sqrt{2}}. \end{aligned}$$

From (15) and (1) we see that

$$(17) \qquad \begin{aligned} a_{-K} &= \frac{\alpha_{-K} + \gamma_{-K}}{\sqrt{2}} = \frac{\gamma_K{}^+ + \alpha_K{}^+}{\sqrt{2}} = a_K{}^+, \\ c_{-K} &= \frac{\alpha_{-K} - \gamma_{-K}}{i\sqrt{2}} = \frac{\gamma_K{}^+ - \alpha_K{}^+}{i\sqrt{2}} = c_K{}^+. \end{aligned} \qquad K > 0$$

[8] P. Jordan and E. Wigner, *Zeits. f. Phy.*, 1928, 47, 631.

$$
\text{(13)}\quad \kappa > 0 \qquad
\begin{array}{l|l}
 & \tfrac{1}{2},\ -\tfrac{1}{2},\ \tfrac{3}{2},\ -\tfrac{3}{2},\ --,\ (\kappa-1),\ -(\kappa-1),\ \kappa,\ -\kappa,\ (\kappa+1),\ -(\kappa+1),\ -- \\
\hline
a_\kappa = \gamma_{-\kappa}^{+} & s \times s \times s \times s \times -- \times s \times s \times a \times 1 \times 1 \times 1 \times -- \\
a_{-\kappa} = \gamma_\kappa^{+} & s \times s \times s \times s \times -- \times s \times s \times s \times a^{+} \times 1 \times 1 \times -- \\
\gamma_\kappa = a_{-\kappa}^{+} & s \times s \times s \times s \times -- \times s \times s \times s \times a \times 1 \times 1 \times -- \\
\gamma_{-\kappa} = a_\kappa^{+} & s \times s \times s \times s \times -- \times s \times s \times a^{+} \times 1 \times 1 \times 1 \times --
\end{array}
$$

Each matrix has only one non-vanishing element given by the following scheme in which t_κ is equal to zero or unity.

$$
\text{(14)}\quad \kappa > 0 \qquad
\left\{
\begin{array}{l}
a_\kappa\ (t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, 0, t_{-\kappa}, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, 1, t_{-\kappa}, --) = (-)^{t_{\frac{1}{2}}+t_{-\frac{1}{2}}+--+t_{-(\kappa-1)}} \\
a_{-\kappa}\ (t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, 1, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, 0, --) = (-)^{t_{\frac{1}{2}}+t_{-\frac{1}{2}}+--+t_\kappa} \\
\gamma_\kappa\ (t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, 0, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, t_\kappa, 1, --) = (-)^{t_{\frac{1}{2}}+t_{-\frac{1}{2}}+--+t_\kappa} \\
\gamma_{-\kappa}\ (t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, 1, t_{-\kappa}, --; t_{\frac{1}{2}}, t_{-\frac{1}{2}}, --, 0, t_{-\kappa}, --) = (-)^{t_{\frac{1}{2}}+t_{-\frac{1}{2}}+--+t_{-(\kappa-1)}}
\end{array}
\right.
$$

From (15) and (3), we obtain the commutation rules for the operators a_κ and c_κ.

$$
(18) \qquad \begin{aligned}
a_\kappa a_\mu + a_\mu a_\kappa &= \delta_{\mu,-\kappa}\,; \\
c_\kappa c_\mu + c_\mu c_\kappa &= \delta_{\mu,-\kappa}\,; \\
a_\kappa c_\mu + c_\mu a_\kappa &= 0\,;
\end{aligned}
$$

for $\kappa, \mu = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \cdots$

As the new operators a_κ and c_κ obey the commutation rules (18), a Jordan-Wigner representation can be developed for them. Thus $a_{-\kappa}\, a_\kappa$ or $a_\kappa^+ a_\kappa$ has the eigenvalues 0 and unity. Similar is the case of $c_{-\kappa}\, c_\kappa$ or $c_\kappa^+ c_\kappa$. We define operators connected with these as

$$
(19) \qquad \begin{aligned}
L_\kappa &= a_\kappa^+ a_\kappa = a_{-\kappa}\, a_\kappa\,; \\
N_\kappa &= c_\kappa^+ c_\kappa = c_{-\kappa}\, c_\kappa\,;
\end{aligned}
$$

for $\kappa > 0$.

From (15) and (19) we find easily,

$$
(20) \qquad \begin{aligned}
L_\kappa &= \tfrac{1}{2}\,(N_\kappa^{(+)} + N_\kappa^{(-)}) + \tfrac{1}{2}\,(a_\kappa^+ \gamma_\kappa + \gamma_\kappa^+ a_\kappa), \\
N_\kappa &= \tfrac{1}{2}\,(N_\kappa^{(+)} + N_\kappa^{(-)}) - \tfrac{1}{2}\,(a_\kappa^+ \gamma_\kappa + \gamma_\kappa^+ a_\kappa),
\end{aligned}
$$

or

$$
(21) \qquad \begin{aligned}
L_\kappa + N_\kappa &= N_\kappa^{(+)} + N_\kappa^{(-)}, \\
L_\kappa - N_\kappa &= a_\kappa^+ \gamma_\kappa + \gamma_\kappa^+ a_\kappa.
\end{aligned}
$$

From the representation (13) of the a_κ, γ_κ one sees that $L_\kappa - N_\kappa$ has no diagonal elements, or the expectation value

$$
(22) \qquad \overline{L_\kappa - N_\kappa} = 0.
$$

$N_\kappa^{(+)}$ and $N_\kappa^{(-)}$ are the numbers of neutrinos and anti-neutrinos in the state κ. We can assume that all states above a certain state are unoccupied,

i.e., $\qquad N_\kappa^{(+)} = 0 \quad$ and $\quad N_\kappa^{(-)} = 0, \quad$ for $\kappa > K$.

Therefore

$$
(23) \qquad \sum_\kappa (L_\kappa + N_\kappa) \text{ is finite.}
$$

From (22) it follows, that all eigenvalues of

$$
(23) \qquad \sum_\kappa (\bar{L}_\kappa - \bar{N}_\kappa) = \overline{\sum_\kappa (L_\kappa - N_\kappa)}
$$

are finite. Therefore

$$
(24) \qquad \sum_\kappa L_\kappa \quad \text{and} \quad \sum_\kappa N_\kappa \text{ are finite}\,; \qquad \kappa > 0.
$$

We shall make use of these results later.

With the help of the operators a_κ and c_κ we will presently define the important operator b_k first discovered by Jordan which obeys the Bose-Einstein commutation rules.

8. *The Operator* b_k.

Jordan has defined the operator b_k as

$$\boxed{b_k = \frac{i}{\sqrt{|k|}} \sum_{\kappa=-\infty}^{\infty} a_\kappa c_{k-\kappa}} \tag{25}$$

for $k = \pm 1, \pm 2, \pm 3, \cdots\cdots$

We first observe that

$$b_k^+ = b_{-k} \tag{26}$$

for

$$b_k^+ = -\frac{i}{\sqrt{|k|}} \sum c_{k-\kappa}^+ a_\kappa^+ = \frac{i}{\sqrt{|k|}} \sum_{-\infty}^{\infty} a_\kappa^+ c^+_{k-\kappa}$$

$$= \frac{i}{\sqrt{|k|}} \sum_{-\infty}^{\infty} a_{-\kappa} c_{-k+\kappa} = \frac{i}{\sqrt{|k|}} \sum_{-\infty}^{\infty} a_\kappa c_{-k-\kappa} = b_{-k}$$

We will now obtain the commutation rules for b_k.

$$[b_k, b_j] = (b_k b_j - b_j b_k) = -\frac{1}{\sqrt{|kj|}} \sum_{\kappa,\iota} [a_\kappa c_{k-\kappa}, a_\iota c_{j-\iota}],$$

$$[a_\kappa c_{k-\kappa}, a_\iota c_{j-\iota}] = a_\kappa [c_{k-\kappa}, a_\iota c_{j-\iota}] + [a_\kappa, a_\iota c_{j-\iota}] c_{k-\kappa}$$

$$= - a_\kappa a_\iota \{c_{k-\kappa}, c_{j-\iota}\} + \{a_\kappa, a_\iota\} c_{j-\iota} c_{k-\kappa}$$

$$= - a_\kappa a_\iota \delta_{k-\kappa, -j+\iota} + c_{j-\iota} c_{k-\kappa} \delta_{\kappa, -\iota}. \tag{27}$$

Thus

$$[b_k, b_j] = \frac{1}{\sqrt{|kj|}} \sum_{-\infty}^{\infty} (a_\kappa a_{k+j-\kappa} - c_{j+\kappa} c_{k-\kappa}). \tag{28}$$

To evaluate the series we remember the result (24) that

$$\sum_\kappa L_\kappa = \sum_\kappa a_{-\kappa} a_\kappa \quad \text{and} \quad \sum_\kappa N_\kappa = \sum_\kappa c_{-\kappa} c_\kappa \tag{29}$$

are convergent. We further assume that the series

$$\sum_{\frac{1}{2}}^{\infty} a_{-\kappa+r} a_\kappa \quad \text{and} \quad \sum_{\frac{1}{2}}^{\infty} c_{-\kappa+r} c_\kappa \tag{30}$$

for any finite r, are convergent.*

Now we reorder the expression (28) in such a way that only terms of the form (30) appear:

$$\sqrt{|kj|}\,[b_k, b_j] = -\sum_{\frac{1}{2}}^{\infty} a_{k+j-\kappa} a_\kappa + \sum_{\frac{1}{2}}^{\infty} c_{k-\kappa} c_{j+\kappa} +$$

* This condition which has not been explicitly stated by Jordan, but which he really uses, seems to us to be indispensable.

$$+ \sum_{-\frac{1}{2}}^{-\infty} a_{\kappa}\, a_{k+j-\kappa} - \sum_{-\frac{1}{2}}^{\infty} c_{j+\kappa}\, c_{k-\kappa}$$

$$= - \sum_{\frac{1}{2}}^{\infty} a_{k+j-\kappa}\, a_{\kappa} + \sum_{\frac{1}{2}}^{\infty} c_{k-\kappa}\, c_{j+\kappa} + \sum_{\frac{1}{2}}^{\infty} a_{-\kappa}\, a_{k+j+\kappa} - \sum_{\frac{1}{2}}^{\infty} c_{j-\kappa}\, c_{k+\kappa}.$$

Replacing in the third series $-\kappa$ by $k + j - \kappa$ and in the fourth series $-\kappa$ by $k - j - \kappa$, we get

$$\sqrt{|k\,j|}\,[b_k, b_j] = - \sum_{\frac{1}{2}}^{\infty} a_{k+j-\kappa}\, a_{\kappa} + \sum_{\frac{1}{2}}^{\infty} c_{k-\kappa}\, c_{j+\kappa}$$

$$+ \sum_{k+j+\frac{1}{2}}^{\infty} a_{k+j-\kappa}\, a_{\kappa} - \sum_{k-j+\frac{1}{2}}^{\infty} c_{k-\kappa}\, c_{j+\kappa} \tag{32}$$

or

$$= - \sum_{\frac{1}{2}}^{k+j-\frac{1}{2}} a_{k+j-\kappa}\, a_{\kappa} + \sum_{\frac{1}{2}}^{k-j-\frac{1}{2}} c_{k-\kappa}\, c_{j+\kappa} = \tag{33}$$

$$- \tfrac{1}{2} \sum_{\frac{1}{2}}^{k+j-\frac{1}{2}} (a_{k+j-\kappa}\, a_{\kappa} + a_{\kappa}\, a_{k+j-\kappa}) + \tfrac{1}{2} \sum_{\frac{1}{2}}^{k-j-\frac{1}{2}} (c_{k-\kappa}\, c_{j+\kappa} + c_{j+\kappa}\, c_{k-\kappa}).$$

Thus using the commutation rules (18)

$$\sqrt{|k\,j|}\,[b_k, b_j] = 0, \qquad \text{if } k + j \neq 0.$$

If $k + j = 0$

$$k\,[b_k, b_{-k}] = \tfrac{1}{2} \sum_{\frac{1}{2}}^{2k-\frac{1}{2}} (c_{k-\kappa}\, c_{-k+\kappa} + c_{-k+\kappa}\, c_{k-\kappa}) = \tfrac{1}{2}\cdot 2k$$

$$= k. \tag{34}$$

Thus we get the commutation rules for b_k as

$$\begin{aligned} b_k\, b_j - b_j\, b_k &= 0, \qquad \text{if } k + j \neq 0, \\ b_k\, b_{-k} - b_{-k}\, b_k &= 1, \qquad k > 0. \end{aligned} \tag{35}$$

9. *Expression for b_k in Terms of α_κ and γ_κ.*

If we substitute in the expression (25) for b_k, α_κ, γ_κ instead of a_κ, c_κ with the help of (15) we get

$$\sqrt{|k|}\, b_k = \tfrac{1}{2} \sum_{-\infty}^{\infty} (\alpha_\kappa + \gamma_\kappa)\,(\alpha_{k-\kappa} - \gamma_{k-\kappa})$$

$$= \tfrac{1}{2} \sum_{-\infty}^{\infty} (\alpha_\kappa\, \alpha_{k-\kappa} - \gamma_\kappa\, \gamma_{k-\kappa} - \alpha_\kappa\, \gamma_{k-\kappa} + \gamma_\kappa\, \alpha_{k-\kappa}).$$

But

$$\sum_{-\infty}^{\infty} \alpha_\kappa\, \alpha_{k-\kappa} = \tfrac{1}{2} \sum_{-\infty}^{\infty} (\alpha_\kappa\, \alpha_{k-\kappa} + \alpha_{k-\kappa}\, \alpha_\kappa) = 0$$

$$\sum_{-\infty}^{\infty} \gamma_\kappa \gamma_{k-\kappa} = \tfrac{1}{2} \sum_{-\infty}^{\infty} (\gamma_\kappa \gamma_{k-\kappa} + \gamma_{k-\kappa} \gamma_\kappa) = 0$$

and
$$\sum_{-\infty}^{\infty} \gamma_\kappa \alpha_{k-\kappa} = -\sum_{-\infty}^{\infty} \alpha_{k-\kappa} \gamma_\kappa = -\sum_{-\infty}^{\infty} \alpha_{k+\kappa} \gamma_{-\kappa} = -\sum_{-\infty}^{\infty} \alpha_\kappa \gamma_{k-\kappa}.$$

Thus we get

$$\boxed{b_k = -\frac{1}{\sqrt{|k|}} \sum_{-\infty}^{\infty} \alpha_\kappa \gamma_{k-\kappa}} \tag{36}$$

or

$$\boxed{b_k = -\frac{1}{\sqrt{|k|}} \sum_{-\infty}^{\infty} \alpha_\kappa \alpha^+_{\kappa-k}} \tag{37}$$

A third expression given by Jordan is gained by splitting the above sum into terms with only positive indices :

$$\boxed{b_k = -\frac{1}{\sqrt{|k|}} \left\{ \sum_{\kappa=\frac{1}{2}}^{k-\frac{1}{2}} \alpha_\kappa \gamma_{k-\kappa} + \sum_{\kappa=\frac{1}{2}}^{\infty} (\alpha_{k+\kappa} \alpha_\kappa^+ - \gamma_{k+\kappa} \gamma_\kappa^+) \right\}} \tag{38}$$

10. *The Operator B and its Commutation Properties with b_k.*

The operator B is defined by Kronig as

$$B = i \sum_{-\infty}^{\infty} a_l c_{-l} \tag{39}$$

and by Jordan as

$$\begin{aligned} B &= \sum_{\frac{1}{2}}^{\infty} (N_\kappa^{(+)} - N_\kappa^{(-)}) \\ &= \sum_{\frac{1}{2}}^{\infty} (\alpha_\kappa^+ \alpha_\kappa - \gamma_\kappa^+ \gamma_\kappa). \end{aligned} \tag{40}$$

The two definitions can be shown to be equivalent if we transform a's and c's to α's and γ's and by using the commutation rules. The first definition in (40) shows immediately (as pointed out by Jordan) that B has whole number eigenvalues $0 \quad \pm 1, \ \pm 2, \ \ldots, \ldots$

Kronig has shown that B commutes with all b_k defined in (25). We simply reproduce his proof.

$$Bb_k = -\frac{1}{\sqrt{|k|}} \sum_{l,m \atop -\infty}^{\infty} a_l c_{-l} a_m c_{k-m},$$

$$\mathrm{B}b_k - b_k\mathrm{B} = \frac{1}{\sqrt{|k|}} \sum_{l,m \atop -\infty}^{\infty} (a_m\, c_{k-m}\, a_l\, c_{-l} - a_l\, c_{-l}\, a_m\, c_{k-m})$$

$$(41) \qquad = \frac{1}{\sqrt{|k|}} \sum_{l \atop -\infty}^{\infty} (a_l\, a_{k-l} + c_{-l}\, c_{k+l}) = 0,$$

using the commutation rules for a's and c's.

11. Photons.

It is known that the photons in a Hohlraum can be represented by a set of variables b_k which satisfy commutation rules of the type (35). In the case of a one-dimensional Hohlraum the light wave could be represented by a wave function

$$(42) \qquad \phi\,(t - x/c) = \sum_{k=-\infty}^{\infty} b_k\, e^{2\pi i \nu_1 k(t-x/c)} = \phi^+.$$

In 3-dimensions there are some geometrical complications arising from the polarisation of light which do not affect the use of the variables* b_k but we do not treat these questions here.

Before Jordan's work it was generally believed that there were two fundamental statistics, the Fermi-Dirac statistics and the Bose-Einstein statistics. The former is connected with wave functions anti-symmetric in the particles of the assembly, the latter with symmetric wave functions. *Jordan's mathematical results can be interpreted as the fundamental physical statement that the only primary statistics is the Fermi-Dirac one with anti-symmetric wave functions.* If applied to processes in empty space, this leads to the idea that photons (as primary Bose-Einstein particles) do not exist at all, but that they are only a secondary effect appearing under special circumstances. The phenomena in empty space have really to be described by moving neutrinos and anti-neutrinos (or the corresponding waves). The phenomena attributed usually to photons are, according to Jordan, really simultaneous actions of pairs of neutrinos and anti-neutrinos. The exact formulation of these actions follows from the formulæ (25).

For a state with given numbers of neutrinos $\mathrm{N}^{(+)}$ and $\mathrm{N}^{(-)}$, the operator representing the *number of photons* in the kth state.

$$(43) \qquad \mathrm{P}_k = b_k{}^+\, b_k = b_{-k}\, b_k,$$

is in general not a diagonal matrix, *i.e.*, the number of photons has not definite values. But we can calculate its average or expectation value—the diagonal element of P_k:

$$\overline{\mathrm{P}}_k\,(t_{1/2},\, t_{-1/2},\, t_{3/2},\, t_{-3/2},\, \cdots\cdots,\, \cdots\cdots)$$

* P. A. M. Dirac, *Quantum Mechanics,* 1935, p. 229.

$$(44) \quad = \sum_{t'_{1/2}, t'_{-1/2}, t'_{3/2}, t'_{-3/2}, \cdots} | b_k (t'_{1/2}, t'_{-1/2}, t'_{3/2}, t'_{-3/2}, \cdots ; \quad t_{1/2}, t_{-1/2}, t_{3/2}, t_{-3/2}, \cdots) |^2.$$

We observe that each a_κ has only one non-vanishing element ± 1, when t'_κ makes the jump $0 \rightarrow 1$ and when all the other t's remain unchanged ; a_κ^+ behaves in the same fashion except that the jump for t'_κ would be $1 \rightarrow 0$. The γ_κ behaves just in a similar way round as indicated in the scheme (14). Therefore $a_\kappa \gamma_{k-\kappa}$ for $\frac{1}{2} \leqslant \kappa \leqslant k - \frac{1}{2}$ has only one non-vanishing element ± 1 for $t_\kappa : 0 \rightarrow 1$ and $t_{\kappa-k} : 0 \rightarrow 1$

One term of the first sum is consequently

$$N_\kappa^{(+)} N_{k-\kappa}^{(-)}.$$

In the same way the terms of the other sums are

$$N_{k+\kappa}^{(+)} (1 - N_\kappa^{(+)}) \text{ and } N_{k+\kappa}^{(-)} (1 - N_\kappa^{(-)}).$$

The result is :

$$(45) \quad \overline{P_k} = \frac{1}{k} \sum_{\kappa=\frac{1}{2}}^{k-\frac{1}{2}} N_\kappa^{(+)} N_{k-\kappa}^{(-)} + \frac{1}{k} \sum_{\kappa=\frac{1}{2}}^{\infty} \{N_{k+\kappa}^{(+)} (1 - N_\kappa^{(+)}) + N_{k+\kappa}^{(-)} (1 - N_\kappa^{(-)})\}.$$

Jordan interprets this formula in the following way : There are two kinds of interaction of neutrinos and matter which are equivalent to an interaction of photons with matter :

(1) *Simultaneous absorption of a neutrino and an anti-neutrino whose total energy is equal to that of the energy change of the atom* : $\kappa + (k - \kappa) = k$ *(usually ascribed to the absorption of one photon).*

(2) *Raman effect of neutrinos or anti-neutrinos* : *One neutrino (anti-neutrino) of the energy* $k + \kappa$ *is absorbed, another of the same kind of the energy* κ *is emitted.*

The processes (1) *and* (2) *are* not *essentially different, if we use the idea of only one kind of neutrinos having positive and negative energy states* ; the process (1) could be considered also as a Raman effect where a neutrino of positive energy is absorbed and one of negative energy is emitted. This is mathematically represented directly by the short formula (37).

12. *Statistical Equilibrium.*

The most important result of Jordan is the proof that this new conception of radiation leads to the correct results for the statistical equilibrium, *i.e.*, Planck's law.

We have two kinds of neutrinos satisfying the Fermi-Dirac statistics. The well-known formula for the entropy of an assembly of $N_l^{(+)}$ neutrinos and $N_l^{(-)}$ anti-neutrinos of nearly equal energy $\overline{l}$ (in units $h\nu_1$) is

$$(46) \quad S = -k \sum_{\mu} \left\{ \begin{array}{l} (1-N_{\mu}^{(+)}) \log (1-N_{\mu}^{(+)}) + N_{\mu}^{(+)} \log N_{\mu}^{(+)} \\ + (1-N_{\mu}^{(-)}) \log (1-N_{\mu}^{(-)}) + N_{\mu}^{(-)} \log N_{\mu}^{(-)} \end{array} \right\},$$

k being the Boltzmann constant.

Since the anti-neutrinos correspond to holes in the multitude of neutrinos with negative energy, a neutrino can disappear only by falling in a hole, or by a simultaneous disappearance of an anti-neutrino. Therefore

$$(47) \qquad \sum_{\mu} (N_{\mu}^{(+)} - N_{\mu}^{(-)}) = B,$$

is constant. Besides we have the total energy given as

$$(48) \qquad \sum_{\mu} \mu\, (N_{\mu}^{(+)} + N_{\mu}^{(-)}) = \frac{E}{h\nu_1}.$$

Using two multipliers which we call $\psi\beta$ and β, we get by the variation of S given by (46) under the two conditions (47) and (48)

$$\log \frac{1 - N_l^{(+)}}{N_l^{(+)}} = \beta\, (l + \psi),$$

$$\log \frac{1 - N_l^{(-)}}{N_l^{(-)}} = \beta\, (l - \psi).$$

From this we find the Fermi distribution law

$$(49) \quad N_l^{(+)} = \frac{1}{e^{\beta(l+\psi)} + 1}, \; N_l^{(-)} = \frac{1}{e^{\beta(l-\psi)} + 1}$$

or if we introduce the notations

$$y = e^{-\beta l} \quad \text{and} \quad a = e^{-\beta\psi},$$

$$(50) \quad N_l^{(+)} = \frac{a\, y}{1 + a\, y} \text{ and } N_l^{(-)} = \frac{\frac{y}{a}}{1 + \frac{y}{a}}.$$

Introducing these into the expression of the entropy and applying the second law of thermodynamics it is shown in the well-known manner that $\beta = h\nu_1/kT$. We substitute these expressions (50) in Jordan's formula (45),

$$\overline{P_l} = \frac{1}{l} \sum_{\kappa=\frac{1}{2}}^{l-\frac{1}{2}} N_{\kappa}^{(+)} N_{l-\kappa}^{(-)} + \frac{1}{l} \sum_{\kappa=\frac{1}{2}}^{\infty} \{N_{l+\kappa}^{(+)} (1 - N_{\kappa}^{(+)}) + N_{l+\kappa}^{(-)} (1 - N_{\kappa}^{(-)})\}.$$

Denoting

$$\frac{\kappa}{l} = \omega,$$

we have

$$e^{-\beta\kappa} = y^{\omega}$$

and $d\kappa = l d\omega$, $\nu_1 l = \nu$.

We replace the sum by integrals* and get

$$(51) \quad \mathrm{P}(\nu) = \int_0^1 \frac{ay^\omega}{1+ay^\omega} \cdot \frac{\frac{1}{a}y^{1-\omega}}{1+\frac{1}{a}y^{1-\omega}}\, d\omega + \int_0^\infty \left\{ \frac{ay^{1+\omega}}{1+ay^{1+\omega}} \cdot \frac{1}{1+ay^\omega} + \frac{\frac{1}{a}y^{1+\omega}}{1+\frac{1}{a}y^{1+\omega}} \cdot \frac{1}{1+\frac{1}{a}y^\omega} \right\} d\omega.$$

The first we write

$$\int_0^1 \frac{y\,d\omega}{(1+ay^\omega)\,(1+\frac{1}{a}y^{1-\omega})} = \int_0^\infty - \int_1^\infty,$$

which by changing in the second integral ω to $1 + \omega$ becomes

$$y \int_0^\infty \left\{ \frac{1}{(1+ay^\omega)\,(1+\frac{1}{a}y^{1-\omega})} - \frac{1}{(1+ay^{1+\omega})\,(1+\frac{1}{a}y^{-\omega})} \right\} d\omega.$$

The second integral here cancels the first part of the remaining integral in (51) so that

$$\overline{\mathrm{P}}(\nu) = y \int_0^\infty \left\{ \frac{1}{(1+ay^\omega)\,(1+\frac{1}{a}y^{1-\omega})} + \frac{\frac{1}{a}y^{1+\omega}}{(1+\frac{1}{a}y^{1+\omega})\,(1+\frac{1}{a}y^\omega)} \right\} d\omega$$

$$= y \int_0^\infty \left\{ \frac{ay^{\omega-1}}{(1+ay^\omega)\,(1+ay^{\omega-1})} + \frac{ay^{-\omega-1}}{(1+ay^{-\omega})\,(1+ay^{-\omega-1})} \right\} d\omega$$

$$= \int_{-\infty}^\infty \frac{ay^\omega}{(1+ay^\omega)\,(1+ay^{\omega-1})}\, d\omega.$$

We write the above formula as

$$(52) \quad \overline{\mathrm{P}}(\nu) = \int_{-\infty}^\infty \frac{ay^\omega}{1+ay^\omega} \left(1 - \frac{ay^{\omega-1}}{1+ay^{\omega-1}}\right) d\omega$$

$$(53) \qquad = \boxed{\frac{1}{l} \sum_{\kappa=-\infty}^{\infty} \mathrm{N}_\kappa \,(1 - \mathrm{N}_{\kappa-l})}$$

---

* One can also readily evaluate these integrals by substituting $\frac{1-z}{z}$ for ay^ω in the first integral and $\frac{1-z}{z}$ for $ay^{1+\omega}$ and $\frac{1-z}{z}$ for $\frac{1}{a}y^{1+\omega}$ in the other integrals.

We can interpret this formula by assuming only one kind of neutrinos with the number N_κ *in the state* κ *where* κ *may be positive or negative.* Then the formula gives the average photon density of energy l as a *Raman effect* with the universal rule: absorption of a neutrino of any (positive or negative) energy κ and simultaneous emission of a neutrino with the energy $\kappa - l$. We mention this to confirm our idea that the introduction of a spin of the neutrino by Jordan is quite unnecessary.

Since $y < 1$ we can substitute

$$ay^{\omega} = \frac{1-z}{z},$$

so that $ay^{\omega}\,d\omega = -\dfrac{1}{z^2 \log y}\,dz.$

Then

$$\overline{P(\nu)} = -\int_0^1 \frac{1}{\log y} \cdot \frac{dz}{z^2\left(1+\frac{1-z}{z}\right)\left(1+\frac{1}{y}\frac{1-z}{z}\right)} = -\frac{y}{\log y}\int_0^1 \frac{dz}{1+(y-1)z}.$$

$$= -\frac{y}{\log y}\left[\frac{\log(1+\overline{y-1}\,z)}{y-1}\right]_0^1 = \frac{y}{1-y}.$$

Introducing here the value of y, we get

$$\overline{P(\nu)} = \frac{e^{-\frac{h\nu}{kT}}}{1-e^{-\frac{h\nu}{kT}}} = \frac{1}{e^{\frac{h\nu}{kT}}-1}. \tag{54}$$

This is the well-known expression for the photon density which by multiplication with the number of photons in the interval $d\nu$ leads to Planck's formula of radiation.

13. *Relation between the Energy of Neutrinos and Photons.*

We have already remarked that a neutrino field with definite numbers of neutrinos of both kinds, $N_\kappa^{(+)}$, $N_\kappa^{(-)}$ is in general not equivalent to a photon field of a given number of photons, P_k. In the language of quantum mechanics we can understand this in the following way. There is a Hilbert space representing all the states of the field. If we choose a co-ordinate system in which the quantities $N_\kappa^{(+)}$, $N_\kappa^{(-)}$ are all represented by diagonal matrices, the quantities P_k are not so. But there will be another co-ordinate system where P_k's are diagonal; we will then have a pure photon state. The question is to form an idea as to the relation of these different states.

For this purpose we prove an identity derived first by de Kronig connecting the total energy of the neutrinos with that of the photons. The energy of the neutrinos is

$$(55) \quad \mathrm{E} = \sum_{n=\frac{1}{2}}^{\infty} n\,(\mathrm{L}_n + \mathrm{N}_n) = \sum_{n=\frac{1}{2}}^{\infty} n\,(\mathrm{N}_n^{(+)} + \mathrm{N}_n^{(-)}).$$

The energy of the photons is

$$(56) \quad \mathrm{W} = \sum_{n=1}^{\infty} n\,\mathrm{P}_n$$

The proposition is

$$(57) \quad \boxed{\mathrm{E} - \mathrm{W} = \tfrac{1}{2}\,\mathrm{B}^2},$$

where B is the quantity defined in (39) which, as we have shown, commutes with all the b_k. Kronig's proof of the proposition is as follows:

$$\begin{aligned} \mathrm{W} &= \sum_{1}^{\infty} k\,\mathrm{P}_k = \sum_{1}^{\infty} k\, b_k^{+}\, b_k \\ &= -\sum_{\substack{l,\,m \\ -\infty}}^{\infty} \sum_{k=1}^{\infty} a_l\, c_{-k-l}\, a_m\, c_{k-m} \\ (58) \quad &= \sum_{\substack{l,\,m \\ -\infty}}^{\infty}{}' \sum_{k=1}^{\infty} a_l\, a_m\, c_{-k-l}\, c_{k-m} + \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} a_l\, a_{-l}\, c_{-k-l}\, c_{k+l}, \end{aligned}$$

where the dash over the summation sign indicates that $l \neq -m$. Changing k to $-k$, l to m, m to l in the first part of (58) and using the commutation laws, we get

$$(59) \quad \mathrm{W} = \sum_{\substack{l,\,m \\ -\infty}}^{\infty}{}' \sum_{k=-1}^{-\infty} a_l\, a_m\, c_{-k-l}\, c_{k-m} + \sum_{l=-\infty}^{\infty} \sum_{k=1}^{\infty} a_l\, a_{-l}\, c_{-k-l}\, c_{k+l}.$$

From (39) we see that

$$(60) \quad \mathrm{B}^2 = \sum_{\substack{l,\,m \\ -\infty}}^{\infty}{}' a_l\, a_m\, c_{-l}\, c_{-m} + \sum_{l=-\infty}^{\infty} a_l\, a_{-l}\, c_{-l}\, c_l$$

We first observe that

$$(61) \quad \sum_{\substack{l,\,m \\ -\infty}}^{\infty}{}' \sum_{k=-\infty}^{\infty} a_l\, a_m\, c_{-k-l}\, c_{k-m} = \sum_{\substack{l,\,m \\ -\infty}}^{\infty}{}' a_l\, a_m \sum_{k=-\infty}^{\infty} c_{-k-l}\, c_{k-m} = 0,$$

using the commutation rules for c's. Thus adding (58), (59) and (60) and using (61), we get

$$2\,\mathrm{W} + \mathrm{B}^2 = 2 \sum_{l=-\infty}^{\infty} a_l\, a_{-l} \sum_{k=1}^{\infty} c_{-k-l}\, c_{k+l} + \sum_{-\infty}^{\infty} a_l\, a_{-l}\, c_{-l}\, c_l.$$

We split the summations in a manner such that each summation index so that k or l assumes only positive values.

$$
\begin{aligned}
2\,\mathrm{W} + \mathrm{B}^2 = {} & 2 \sum_{l=\frac{1}{2}}^{\infty} a_l\, a_{-l} \sum_{k=1}^{\infty} c_{-k-l}\, c_{k+l} + 2 \sum_{l={}^3/_2}^{\infty} a_{-l}\, a_l \sum_{k=1}^{l-\frac{1}{2}} c_{-k+l}\, c_{k-l} \\
& + 2 \sum_{l={}^3/_2}^{\infty} a_{-l}\, a_l \sum_{k=l+\frac{1}{2}}^{\infty} c_{-k+l}\, c_{k-l} + 2\, a_{-\frac{1}{2}}\, a_{\frac{1}{2}} \sum_{1}^{\infty} c_{-k+\frac{1}{2}}\, c_{k-\frac{1}{2}} \\
& + \sum_{l=\frac{1}{2}}^{\infty} a_l\, a_{-l}\; c_{-l}\; c_l + \sum_{l=\frac{1}{2}}^{\infty} a_{-l}\, a_l\, c_l\, c_{-l} \\
= {} & 2 \sum_{l=\frac{1}{2}}^{\infty} (1 - \mathrm{L}_l) \sum_{k=1}^{\infty} \mathrm{N}_{k+l} + \sum_{l={}^3/_2}^{\infty} \mathrm{L}_l \sum_{k=1}^{l-\frac{1}{2}} (1 - \mathrm{N}_{-k+l}) \\
& + 2 \sum_{l={}^3/_2}^{\infty} \mathrm{L}_l \sum_{k=l+\frac{1}{2}}^{\infty} \mathrm{N}_{k-l} + 2\, \mathrm{L}_{\frac{1}{2}} \sum_{k=1}^{\infty} \mathrm{N}_{k-\frac{1}{2}} \\
& + \sum_{l=\frac{1}{2}}^{\infty} (1 - \mathrm{L}_l)\, \mathrm{N}_l + \sum_{l=\frac{1}{2}}^{\infty} \mathrm{L}_l\, (1 - \mathrm{N}_l) \\
= {} & 2 \sum_{l=\frac{1}{2}}^{\infty} (1 - \mathrm{L}_l) \sum_{j=l+1}^{\infty} \mathrm{N}_j + 2 \sum_{l={}^3/_2}^{\infty} \mathrm{L}_l \sum_{j=\frac{1}{2}}^{l-1} (1 - \mathrm{N}_j) \\
& \qquad\qquad + 2 \sum_{l={}^3/_2}^{\infty} \mathrm{L}_l \sum_{k=\frac{1}{2}}^{\infty} \mathrm{N}_j \\
& + 2\, \mathrm{L}_{\frac{1}{2}} \sum_{k=1}^{\infty} \mathrm{N}_{k-\frac{1}{2}} + \sum_{l=\frac{1}{2}}^{\infty} (1 - \mathrm{L}_l)\, \mathrm{N}_l + \sum_{l=\frac{1}{2}}^{\infty} \mathrm{L}_l\, (1 - \mathrm{N}_l) \\
= {} & (\mathrm{L}_{\frac{1}{2}} + \mathrm{N}_{\frac{1}{2}}) + 2 \sum_{l={}^3/_2}^{\infty} \sum_{j=l+1}^{\infty} \mathrm{N}_j + 2 \sum_{l={}^3/_2}^{\infty} \mathrm{L}_l \sum_{j=\frac{1}{2}}^{l-1} 1 \\
& \qquad\qquad + 3 \sum_{{}^3/_2}^{\infty} \mathrm{N}_j + \sum_{{}^3/_2}^{\infty} \mathrm{L}_j \\
= {} & (\mathrm{L}_{\frac{1}{2}} + \mathrm{N}_{\frac{1}{2}}) + 2 \sum_{{}^3/_2}^{\infty} (l - \tfrac{3}{2})\, \mathrm{N}_l + 2 \sum_{{}^3/_2}^{\infty} (l - \tfrac{1}{2})\, \mathrm{L}_l + 3 \sum_{{}^3/_2}^{\infty} \mathrm{N}_l + \sum_{{}^3/_2}^{\infty} \mathrm{L}_l \\
= {} & 2 \sum_{l=\frac{1}{2}}^{\infty} l\, (\mathrm{L}_l + \mathrm{N}_l) = 2\, \mathrm{E}.
\end{aligned}
$$

Thus

$$2\,\mathrm{W} + \mathrm{B}^2 = 2\,\mathrm{E} \tag{62}$$

or

$$\boxed{\mathrm{E} - \mathrm{W} = \mathrm{B}^2/2}$$

Now we can answer the question put forward in the beginning of this section.

The necessary and sufficient condition that the whole neutrino energy appears in the form of photon energy is $B = 0$ or $N^{(+)} = N^{(-)}$. These states are called pure photon states or pure light fields.

The other extreme case is a pure neutrino field if $W = 0$ and $E = B^2/2$.

There are innumerable cases intermediate between these pure fields. As we have shown in (41) that B commutes with all b_k, the matrices representing b_k are reducible; they split up into finite parts corresponding to the eigenvalues $0, \pm 1, \pm 2, \ldots,$ of B. For, we can take a matrix representation where B is diagonal; if r and s symbolise states of the neutrinos, then

$$B(r, s) = B(r)\delta(r, s).$$

The commutability

$$B\, b_k = b_k\, B$$

gives in this representation

$$B(r)\, \delta(r, t)\, b_k(t, s) = b_k(r, t)\, B(t)\, \delta(t, s),$$

or $$B(r)\, b_k(r, s) = b_k(r, s)\, B(s),$$

from which we conclude that $b_k(r, s) = 0$ for each pair of states which do not belong to the same eigenvalue of B, $B(r) \neq B(s)$. Therefore all impure fields can be classified with 0, or 1, or 2, ... not compensated neutrinos (or anti-neutrinos); Jordan calls this number the *resultant neutrino charge.*

14. *Conclusion.*

The standpoint assumed by Jordan is the following. Before the radioactive processes revealed the probability of the existence of the neutrinos, the only experimentally known wave fields were those which appear now as pure light fields. They turn out to be only a limiting case of a very much larger multitude of possible fields, which contain free (not compensated) neutrinos. The validity of this hypothesis could be experimentally tested by a study of a possible influence of radiation fields on the β-decay. If it is true that the β-emission is accompanied by an emission of a neutrino there should be an influence of an external neutrino field on the β-emission. But since light fields are nothing than neutrino fields (with neutrino pairs) we should also expect an influence of light radiation on the β-decay. The law of this interaction has yet to be calculated.

We wish to make another remark. Quantum mechanics was started by replacing the Fourier amplitudes $q_k e^{2\pi i \nu_0 k t}$ of co-ordinate function $q(t)$ by matrix elements with two indices $q_{kl} e^{2\pi i \nu_{kl} t}$. In analogy, one could expect that in a quantum field theory the Fourier elements $q_k e^{2\pi i \nu_0 k(t-x/c)}$ of a quantity representing a progressive wave $q(t - x/c)$ should be replaced by matrix elements $q_{kl} e^{2\pi i \nu_{kl}(t-x/c)}$. Here, as in quantum mechanics, one should expect the combination law $\nu_{kl} + \nu_{lm} = \nu_{km}$. In the previous theories of radiation fields, there was no indication that something like that would lead to a deeper understanding. But here, in Jordan's neutrino theory, we have for the photon amplitudes the expression

$$\frac{\sqrt{|k|}}{i} b_k = \sum_{l=-\infty}^{\infty} a_l \, c_{k-l}, \tag{63}$$

where the right-hand sum is apparently the coefficient of the two Fourier series with the coefficients a_k, c_k. Now it suggests itself that we have really to do with matrices and their multiplication : a_{kl}, c_{kl} and

$$\boxed{\sum_{n=-\infty}^{\infty} a_{kn} \, c_{nl}} \tag{64}$$

where each element of these matrices is again a matrix or more generally a non-commuting quantity of the kind treated here. It seems to be attractive to follow this indication.

During writing this article, we have received the manuscript of a new article* of Jordan and de Kronig in which they treat the 3-dimensional case. There appear no new fundamental difficulties, but there are some interesting features which are beyond the scope of this article to be reported. Also in this paper, Jordan and Kronig have not changed their standpoint to consider the distinction between the neutrino and the anti-neutrino as the value of the spin instead of states and holes as we do here.

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* We are very grateful for having had the privilege of knowing about this work before publication.