Teaching STEM Math to first year college students

Keynote address: `http://huichawaii.org/ssec/overview/`

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Abstract

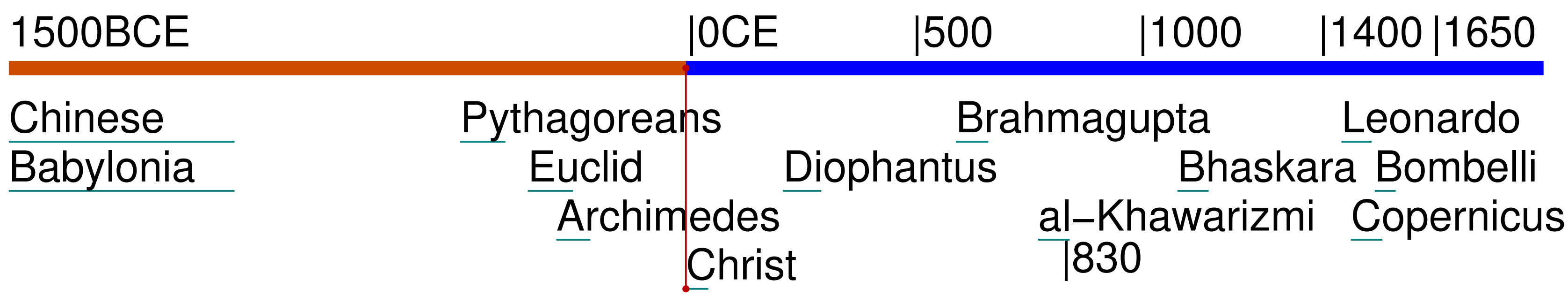
It is widely acknowledged that the goal of STEM is to unify scientific training. To this end, the fundamental theorems of mathematics (arithmetic, algebra, real and complex calculus, linear algebra, vector calculus, etc.) need to be appreciated by every student. At the core of this teaching are 1) linear algebra of several complex variables, 2) complex calculus, and 3) partial differential equations (e.g., Maxwell's Eqs). Understanding the way these early ideas evolved, and were then generalized, provides fundamental insights. For example, the Fundamental theorem of complex calculus (Residue integration) can be viewed as a generalization of the Fundamental theorem of calculus (Leibniz formula). Including the mathematical history provides a uniform terminology for understanding these generalizations. The present teaching methods (abstract proofs with few figures or physical principles), by design, removes intuition and the motivation that was available to the creators of these early theories. The present six-semester approach to mathematics does not function for many students, leaving them with poor, or worse, no intuition. When students are taught a common mathematical language based on the historical context, they are equipped to communicate with other interdisciplinary scientists. The historical perspective is the key to such a unification.

Goals for: "A review of mathematics, via its history."

- First year college students take calculus in high school
- >30–40% advance-place (AP) out of Calc-I & II
  - Due to poor fundamentals, they struggle in engineering courses
  - Mathematics courses (Calc-III, Linear Alg., DiffEq, ...) are a mystery
- Solution: *Concepts in Mathematical–Physics based & its History*
  - Proven to work, and students love it:
    - "I have to wonder why it isn't the standard way of teaching mathematics to engineers."
    - "Fourier series and Laplace transforms and distributions are now understandable, even easy."
    - "Engineering courses are now 'easy' after ECE298ja"
    - "Learning complex analysis makes math less 'magical.' "
    - "Homeworks are hard, but worth the effort"
    - "ECE298ja students are #1 in their Math and Engineering classes
    - "Thank you for teaching 298: It felt like "Whoa!"
    - "I can now keep up with friends at Harvard in the infamous Math 55.

Time-line: 5000 BCE–1650 CE

- Early Chinese: Gaussian elimination; quadratic formula;
- Pythagoreans, Euclid, Diophantus
- Algebra al-Khawarizmi 830 CE
- Bombelli discovers Diophantus' *Arithmetica* in Vatican library



Pythagoras's demise

- Pythagoras was murdered by townspeople:

  Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Metapontum in 497 BC. [Stillwell, 2010, p. 16]

Key concepts in math and physics

- Fundamental theorems
  - Integers may be factored into primes (FT Arith)
  - Density of primes within integers (PNT)
  - Algebra (factoring polynomials)
  - Integral Calculus (Real and complex integration)
  - Vector calculus (Helmholtz Theorem)
- Other key theorems:
  - Complex analytic functions
  - Calculus in the complex plane
  - Cauchy Integral Theorem (Residue integration)
  - Riemann sphere (defining the point at ∞)
- Applications:
  - Linear algebra
  - Difference, scalar & vector differential equations
  - Maxwell's vector differential equations

Generalizations of the Pythagorean Formula

- The circle: $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$
- Complex numbers: $c^2 = |a + b\jmath|^2 = a^2 + b^2$
- *Riemann sphere & surface*, *Branch cuts*
- Three streams flow from the Pythagorean theorem Stillwell [2010]
  - Numbers (e.g., properties of primes)
  - Geometry (geometry+algebra = analytic geometry)
  - Infinity (limits and Calculus: scalars and vectors)

The three Streams and their mathematics Stillwell [2010]

- *The Pythagorean Theorem bore three streams:*
- *2–3 Centuries per stream:*

1) **Numbers:**

6^{th}BCE	$\mathbb{N}$ *counting numbers*, $\mathbb{Q}$ *(Rationals)*, $\mathbb{P}$ *Primes*
5^{th}BCE	$\mathbb{Z}$ *Common Integers*, $\mathbb{I}$ *Irrationals*
7^{th}CE	*zero* $\in \mathbb{Z}$

2) **Geometry:** *(e.g., lines, circles, spheres, toroids, . . .)*

17^{th}CE	*Composition of polynomials (Descartes, Fermat)*
	Euclid's Geometry + algebra $\Rightarrow$ *Analytic Geometry*
18^{th}CE	*Fundamental Theorem of Algebra*

3) **Infinity:** *(*$\infty \rightarrow$ *Sets)*

17-18^{th}CE	*Taylor series, analytic functions, calculus (Newton)*
19^{th}CE	$\mathbb{R}$ *Real*, $\mathbb{C}$ *Complex 1851; Open vs. closed Sets 1874*

Fundamental theorems of:

1. **Number systems: Stream 1**
   - arithmetic (FTA)
   - prime number (PNT)
2. **Geometry: Stream 2**
   - algebra
   - Bézout
3. **Calculus: Stream 3**
   - Leibniz $\mathbb{R}^1$ (area under a curve only depends on end points)
   - complex $\mathbb{C} \subset \mathbb{R}^2$ (area under a curve only depends on end points!)
   - vectors $\mathbb{R}^3, \mathbb{R}^n, \mathbb{R}^\infty$
     - Gauss' Law (Divergence theorem)
     - Stokes' Law (Curl theorem, or Green's theorem)
     - Vector calculus (Helmholtz's theorem)

Stream 1: WEEK 2-10, Lects 2-10

- Stream 1: Numbers (WEEK 2-3, Lects 2-10, Lects 2-10)
- Stream 2: Algebraic Equations (WEEK 4-8, Lect 11-22)
- Stream 3a: Scalar Differential Equations (WEEK 8-12, Lect 11-22)
- Stream 3b: Partial Differential Equations (WEEK 12-14, Lect 11-22)

Famous problems in number theory (Stream 1)

- Finding prime numbers using sieves
- Greatest common divisor [3=gcd(15,6) $\Rightarrow$ 3=15/5]
- Continued fraction algorithm (rational approximations of irrational numbers)
- Pythagorean triplets (integer solutions of $c^2 = a^2 + b^2$)
- Pell's equation (integer solutions of $a^2 - 2b^2 = 1$
- Fibonocci sequence (the next number is the sum of the previous two)

Finding prime numbers: the sieve of Eratosthenes

1. Write N integers from 2 to $N-1$. Set $k=1$. The first element $\pi_1 = 2$ is prime. Cross out $n \cdot \pi_n$: (e.g., $n \cdot \pi_1 = 4, 8, 16, 32, \cdots$).

	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~

2. Set $k=2$, $\pi_2 = 3$. Cross out $n\pi_k$ $(6, 9, 12, 15, \ldots)$:

	2	**3**	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~

3. Set $k=3, \pi_3 = 5$. cross out $n\pi_3$. (Cross out 25, 35).

	2	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~

4. Finally let $k=4, \pi_4 = 7$. Remove $n\pi_4$: (Cross out 49).

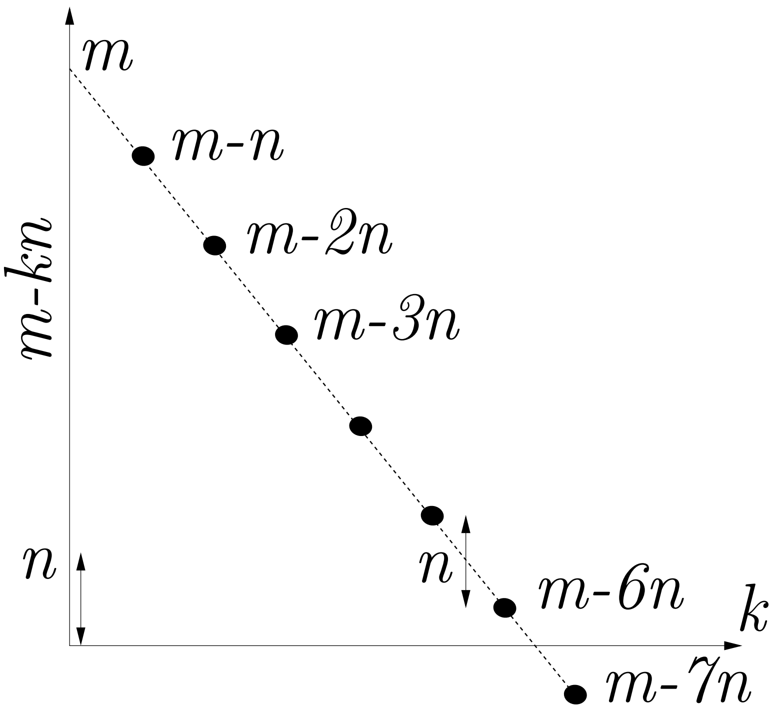
There are 15 primes less than $N = 50$: $\pi_k = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.

Greatest common divisor (factors): $c = gcd(a, b)$

Ex: $17 = gcd(17 \cdot 3, 17 \cdot 5)$ $(a, b, c \in \mathbb{N})$. In matrix form:

$$\begin{bmatrix} m_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\left\lfloor \frac{m}{n} \right\rfloor \end{bmatrix} \begin{bmatrix} m_k \\ n_k \end{bmatrix}. \tag{1}$$

This starts with $k = 0, m_0 = a, n_0 = b$.



The Euclidean Algorithm recursively subtracts n_k from m_k until the remainder $m_k - c_k n_k \le 0$. The GCD recursively computes mod(m,n), then swaps m, n so, $n < m$. This repeats until it finds c=`gcd(a,b)`. Division gives $m/n \approx 6.4$; thus $6 = \left\lfloor \frac{m}{n} \right\rfloor$, leaving remainder $r = m - 6n = 0.4$. Thus

$$n_{n+1} = m_k - \left\lfloor \frac{m}{n} \right\rfloor n_k = m_k - 6n_k < n_k.$$

If one more step were taken the remainder would become negative: $r = m - 7n = -0.6$.

Continued fraction algorithm (cfa)

Given an irrational number $x \in \mathbb{I}$, $n/m = CFA(x)$ finds a rational approximation $n/m \in \mathbb{Q}$, to any desired accuracy.

Examples:

$$\widehat{\pi}_1 \approx 3 + \frac{1}{7 + 0.0625\ldots} \approx 3 + \frac{1}{7} = \frac{22}{7}$$

$$\widehat{\pi}_2 \approx 3 + 1/(7 + 1/16) = 3 + 16/113 = 355/113$$

$$\widehat{e_5} = 3 + 1/(-4 + 1/(2 + 1/(5 + 1/(-2 + 1/(-7))))) - 1.753610^{-6}$$

Pythagorean triplets & Euclid's formula

Find $a, b, c \in \mathbb{N}$ such that

$$c^2 = a^2 + b^2.$$

Solution: Set $p > q \in \mathbb{N}$. Then (Euclid's formula)

$$c = p^2 + q^2, \qquad a = p^2 - q^2, \qquad b = 2pq. \tag{2}$$

This result may be directly verified

$$[p^2 + q^2]^2 = [p^2 - q^2]^2 + [2pq]^2$$

or

$$p^4 + q^4 + \cancel{2p^2q^2} = p^4 + q^4 - \cancel{2p^2q^2} + \cancel{4p^2q^2}.$$

Deriving Euclid's formula (Eq. 2) is obviously more difficult.

Pell's equation $x^2 - Ny^2 = 1,\ N \in \mathbb{N}$

- Solution for the case of $N = 2$ & $[x_o, y_o]^T = [1, 0]^T$
  Solution: $x_n^2 - 2y_n^2 = 1,\qquad x_n/y_n \underset{\infty}{\longrightarrow} \sqrt{2}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \jmath^2 - 2 \cdot \jmath^2 = 1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad (7\jmath)^2 - 2 \cdot (5\jmath)^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \jmath^3 \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad 17^2 - 2 \cdot 12^2 = 1$$

$$\cdots$$

Following each iteration, $x_n/y_n \to \sqrt{2}$ with increasing accuracy, coupling it to the CFA.

Fibonacci sequence $f_{n+1} = f_n + f_{n-1}$

The sequence is

$$f_n = 0, 1, 1, 2, 3, 5, 8, 13, \cdots .$$

Each output is the sum of the last two.
Alternatively we may define $y_{n+1} = x_n$, then the Fibonacci sequence may be represented with the 2x2 matrix recursion:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

The correspondence is easily verified.

Stream 2: WEEK 4, Lects 11-22

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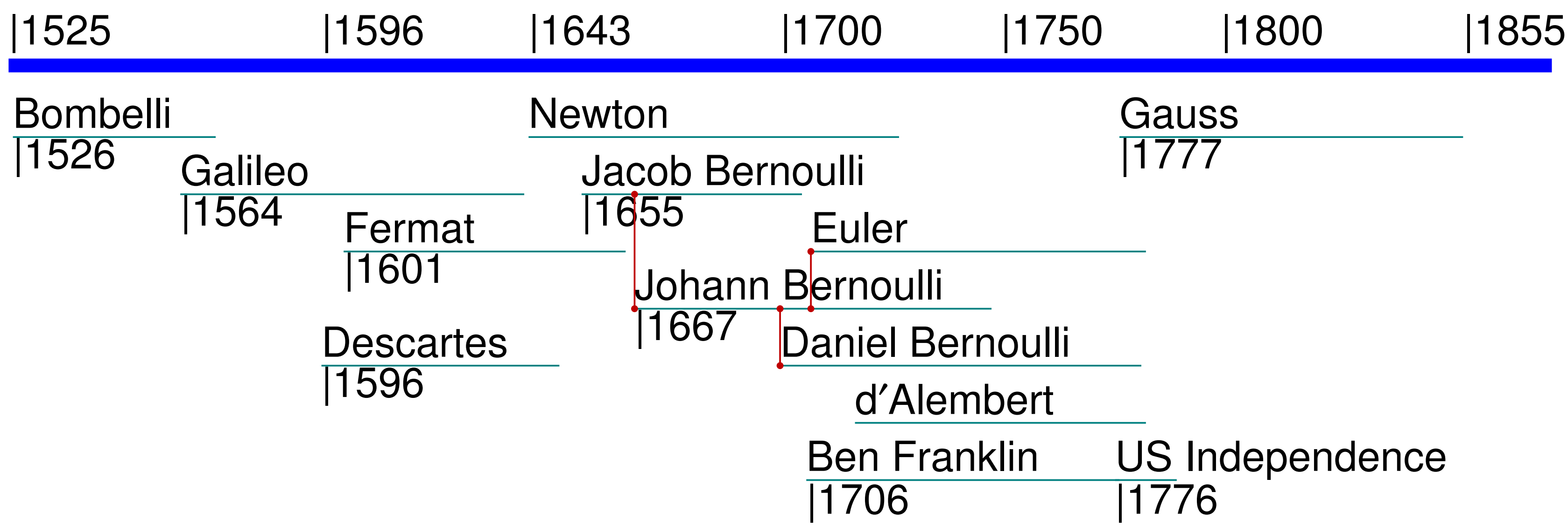
Time-line: Bombelli–Gauss 16-18 centuries



- Newton, Bernoulli family, Euler, d'Alembert and Gauss
- Johann teaches mathematics to Euler and Daniel
- Euler's technique dominates mathematics for 200 years (examples)
- d'Alembert proposes:
  - general solution to scalar wave equation
  - fundamental theorem of algebra (FTA)
- Gauss had great conceptual depth (PNT, Least squares, FFT)

Analytic functions and Taylor series (Newton)

An *analytic function* is one that may be expanded in a power series

- Geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

- This is easily seen to be correct by cross-multiplying

$$1 = (1-x)\sum x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=1}^{\infty} x^n = 1$$

- The *Taylor series* is much more powerful

$$f(x) = \sum_n \underbrace{\frac{1}{n!}\frac{d^n}{dx^n} f(x)}_{a_n}\Bigg|_{x=0} x^n$$

Jakob Bernoulli #1 (1654-1705)

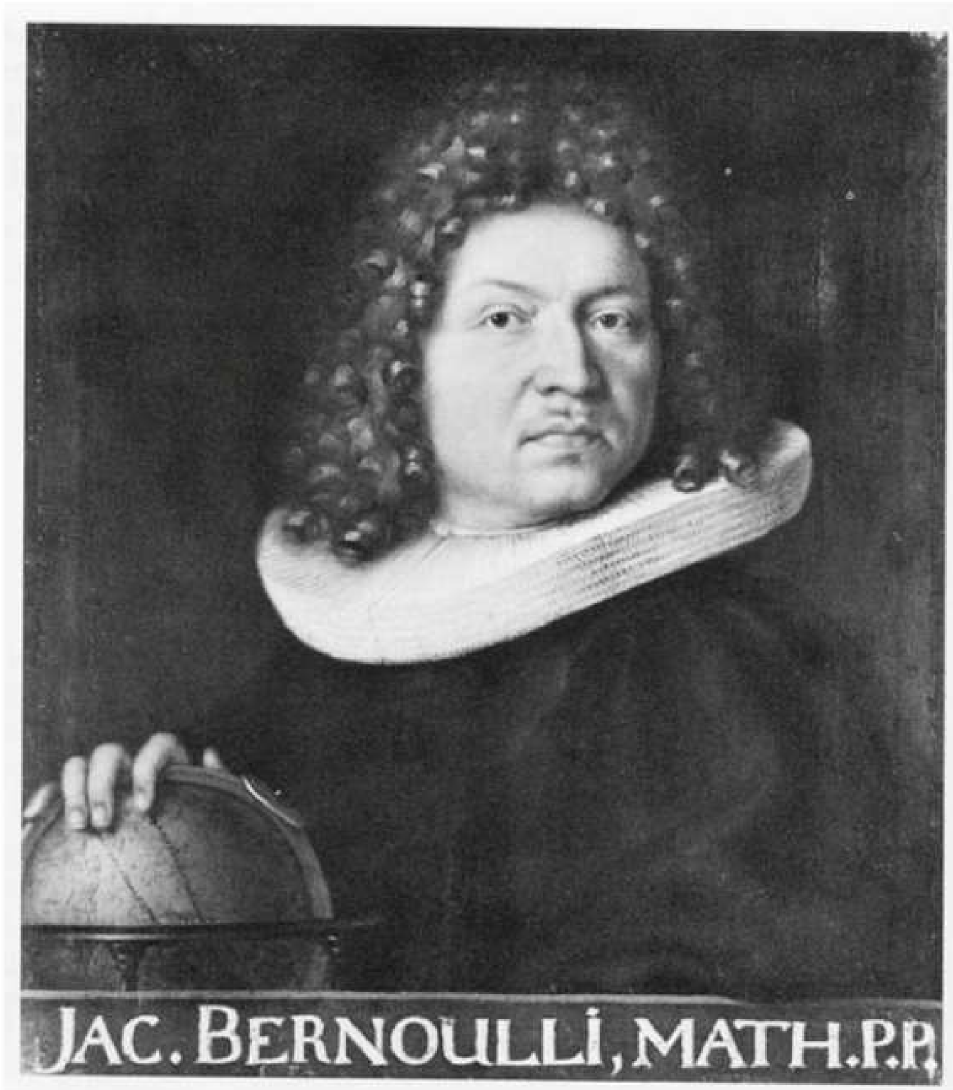


Figure 13.10: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Johann Bernoulli (#2) 10^{th} child; Euler's advisor

Figure 13.11: Johann Bernoulli

Leonhard Euler, most prolific of all mathematicians

Figure 10.4: Leonhard Euler

Euler's sieve and the zeta function: $\zeta(s)$

The Euler's zeta function is an algebraic replica of Eratosthenes sieve

$$\zeta(s) \equiv \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s} \quad \text{for } \Re s = \sigma > 0. \tag{3}$$

Multiplying $\zeta(s)$ by the factor $1/2^s$, and subtracting from $\zeta(s)$, removes all the even terms $\propto 1/(2n)^s$ (e.g., $1/2^s + 1/4^s + 1/6^s + 1/8^s + \cdots$)

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \cancel{\frac{1}{2^s}} + \frac{1}{3^s} + \cancel{\frac{1}{4^s}} + \frac{1}{5^s} \cdots - \left(\cancel{\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s}} + \cdots\right), \tag{4}$$

results in

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \cdots. \tag{5}$$

Likewise

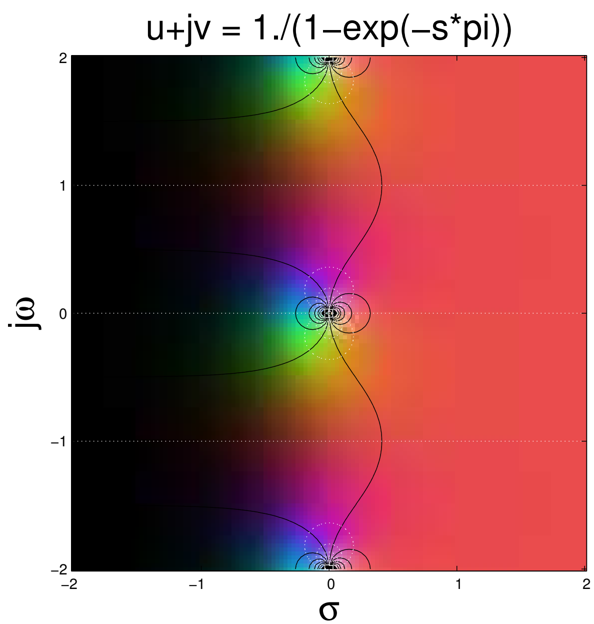
$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \frac{1}{19^s} \cdots.$$

Euler's sieve gives Euler's product formula of $\zeta(s)$

$$\zeta(s) = \prod_{\pi_k \in \mathbb{P}} \frac{1}{1-\pi_k^{-s}} = \prod_{\pi_k \in \mathbb{P}} \zeta_k(s), \tag{6}$$

where π_k represents the k^{th} prime. The above defines each prime factor

$$\zeta_k(s) = \frac{1}{1-\pi_k^{-s}} = \frac{1}{1-e^{-s\ln \pi_k}} \tag{7}$$



Plot of $w(s) = \frac{1}{1-e^{-s\ln \pi_1}}$ $(\pi_1 = 2)$, factor $\zeta_1(s)$ (Eq. 6), which has poles where $e^{s_n \ln 2} = 1$, namely where $\omega_n \ln 2 = n2\pi$, as demonstrated by the domain–color map. $s = \sigma + \omega \jmath$ is the Laplace frequency.

Stream 3a: WEEK 8-12, Lects 23-34

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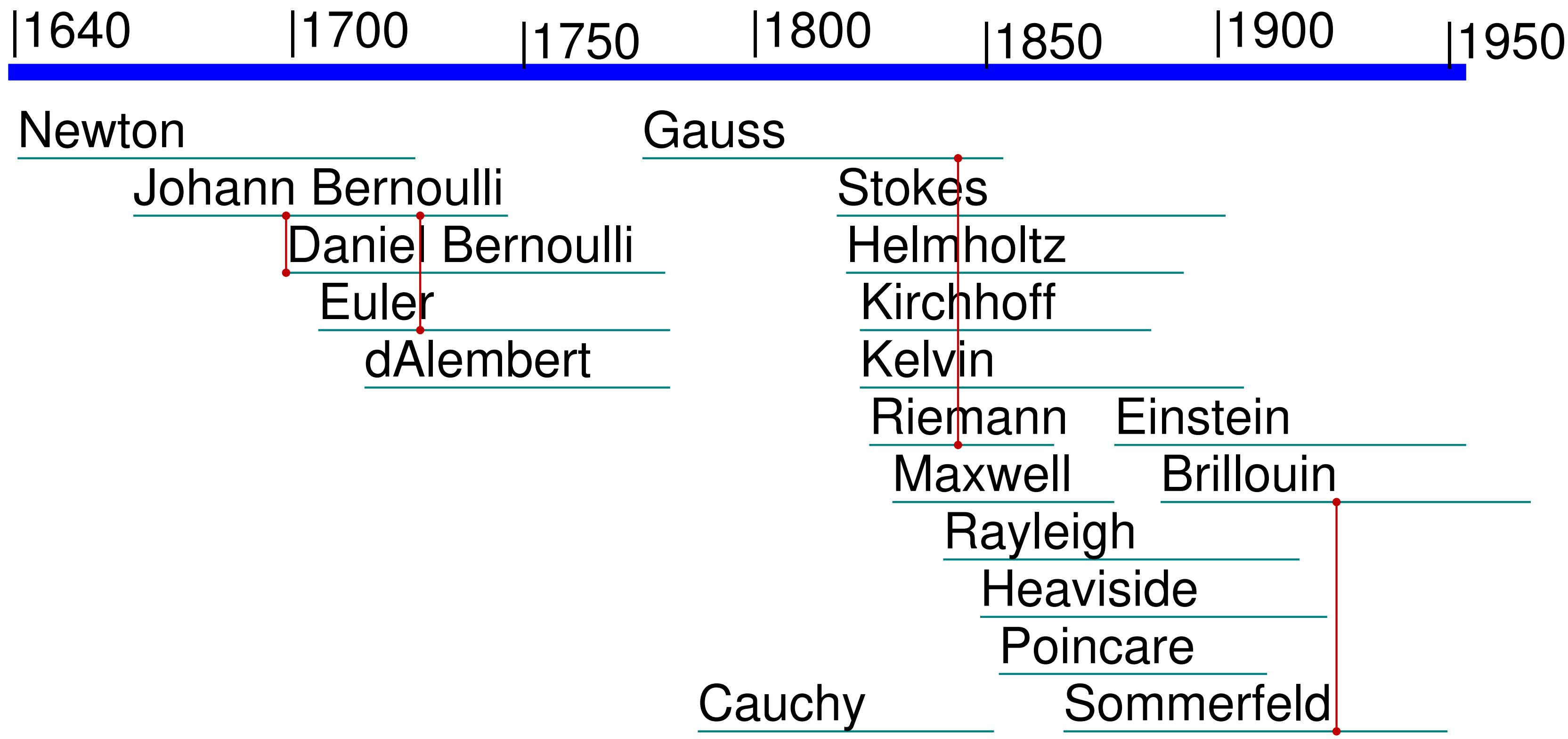
- Stream 3b: Partial Differential Equations (WEEK 12-14, Lect 35-42)

Time-line Newton-Einstein 1640-1950



- Notes:
  - Gaussian gap: Euler $\Rightarrow$ Helmholtz
  - Connection between Gauss & Riemann
  - Heritage: Stokes & Helmholtz $\Rightarrow$ Sommerfeld & Einstein

Complex Analytic functions and Taylor series

An *analytic function* is one that may be expanded in a complex power series. Replace $x \in \mathbb{R}$ with $z = x + iy \in \mathbb{C}$

- Geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n$$

- Cross-multiplying shows this series is correct

$$1 = (1-z)\sum x^n = \sum_{n=0}^{\infty} z^n - \sum_{n=1}^{\infty} z^n = 1$$

- However the more general Taylor series has a problem: $z, F(z) \in \mathbb{C}$

$$F(z) = \sum_n \underbrace{\frac{1}{n!}\frac{d^n}{dz^n} f(z)}_{c_n}\Bigg|_{x=0} z^n$$

What does it mean to differentiate wrt $z \in \mathbb{C}$?

$$\frac{d}{dz}F(z) = \frac{d}{d(x+y\jmath)}F(x+y\jmath)???$$

Complex analytic functions to solve difference equations

- Define *Laplace frequency* $s = \sigma + \omega \jmath$
- If
$$e^{st} = \sum_{n=1}^{\infty} \frac{1}{n!}(st)^n$$
- then
$$\frac{d}{dt}e^{st} = se^{st}$$
  - e^{st} is an *eigenvector* of $\frac{d}{dt}$

Matrix recursion $x_{n+1} = \mathbf{A}x_n$ by eigenvectors

Any power of a matrix $\mathbf{A}$ may be computed from a *matrix eigenvectors*:

- Specifically
$$\mathbf{AE} = \mathbf{E}\Lambda \quad \Leftrightarrow \quad \mathbf{A} = \mathbf{E}\Lambda\mathbf{E}^{-1}. \tag{8}$$
- For example, $x_2 = Ax_1 = A^2x_0$
$$\mathbf{A}^2 = \mathbf{AA} = \mathbf{E}\Lambda\, \mathbf{E}^{-1}\, \mathbf{E}\, \Lambda\mathbf{E}^{-1} = \mathbf{E}\Lambda^2\mathbf{E}^{-1}.$$
- Matrix recursion may be solved via *eigenvector expansion*
$$\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}, \tag{9}$$
greatly simplifying matrix recursion (e.g., Pell, Fibonacci)

d'Alembert: Creative, prolific & respected

Mapping the multi-valued square root of $w = \pm\sqrt{x+iy}$

- This provides a deep (essential) analytic insight.

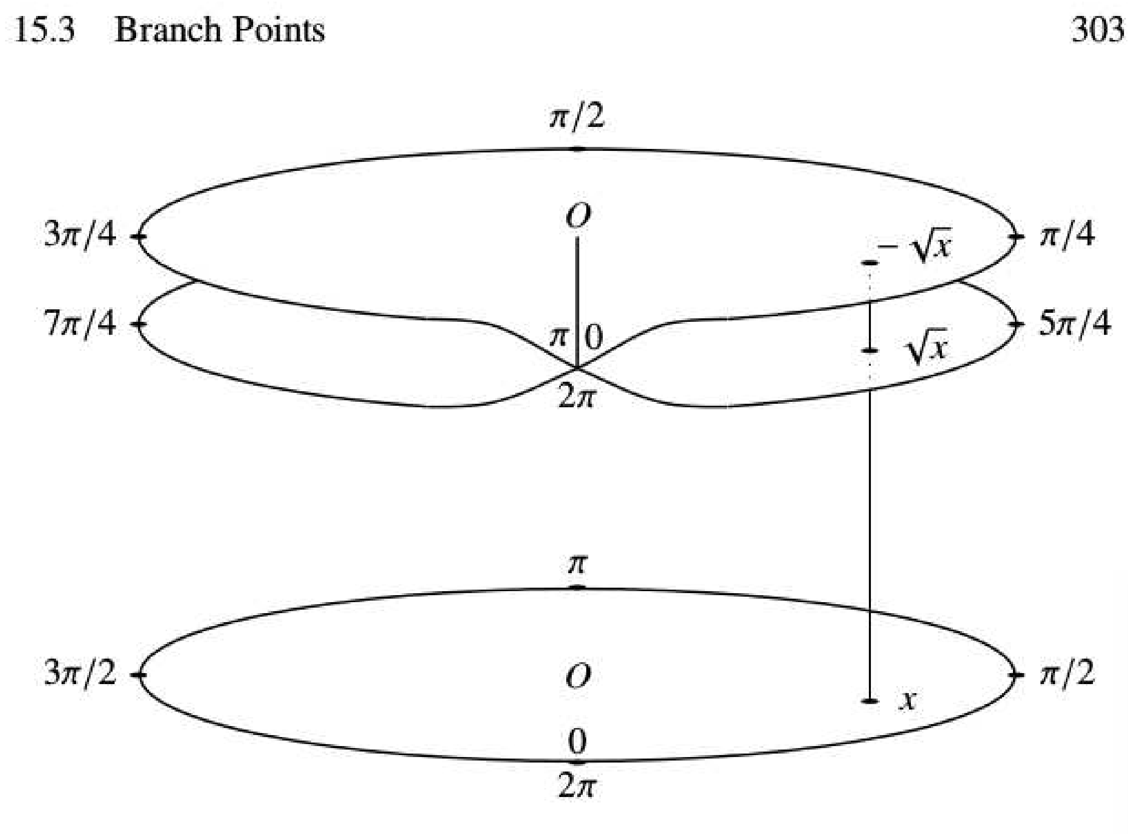
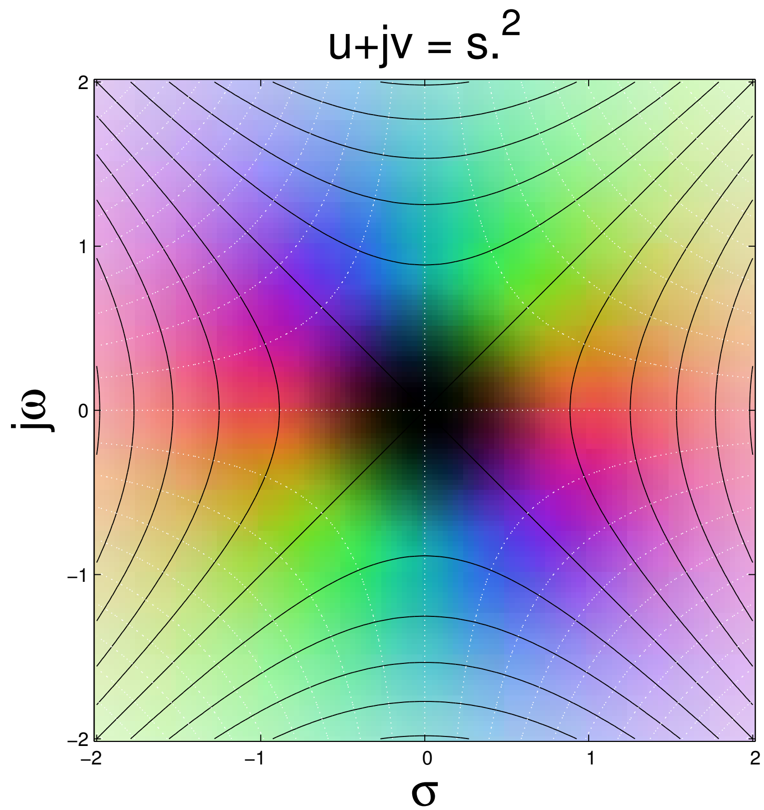


Figure 15.6: Branch point for the square root

- The Riemann Surface of the cubic $y^2 = x(x-a)(x-b)$ has Genis 1 (torus) (p. 307). *Elliptic functions* naturally follow.

Mapping complex analytic function



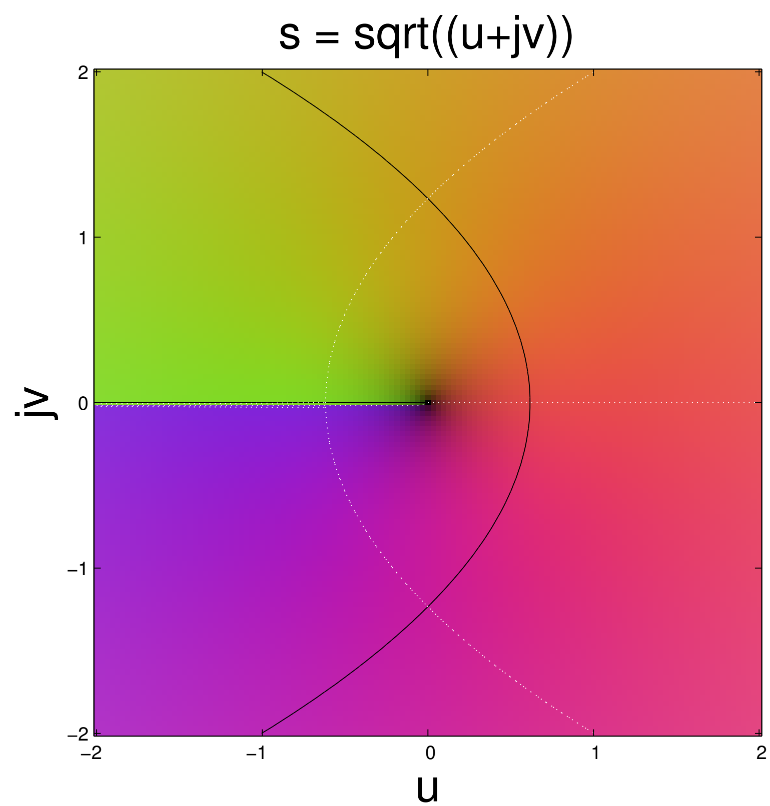


Figure: *Here the Cartesian coordinate map between $s = \sigma + \omega\jmath$ and $w = u + v\jmath$. LEFT: This shows the mapping $w(s) = s^2$. RIGHT: This shows the lower branch of the inverse $s(w) = \sqrt{w}$.*

Mapping complex analytic function

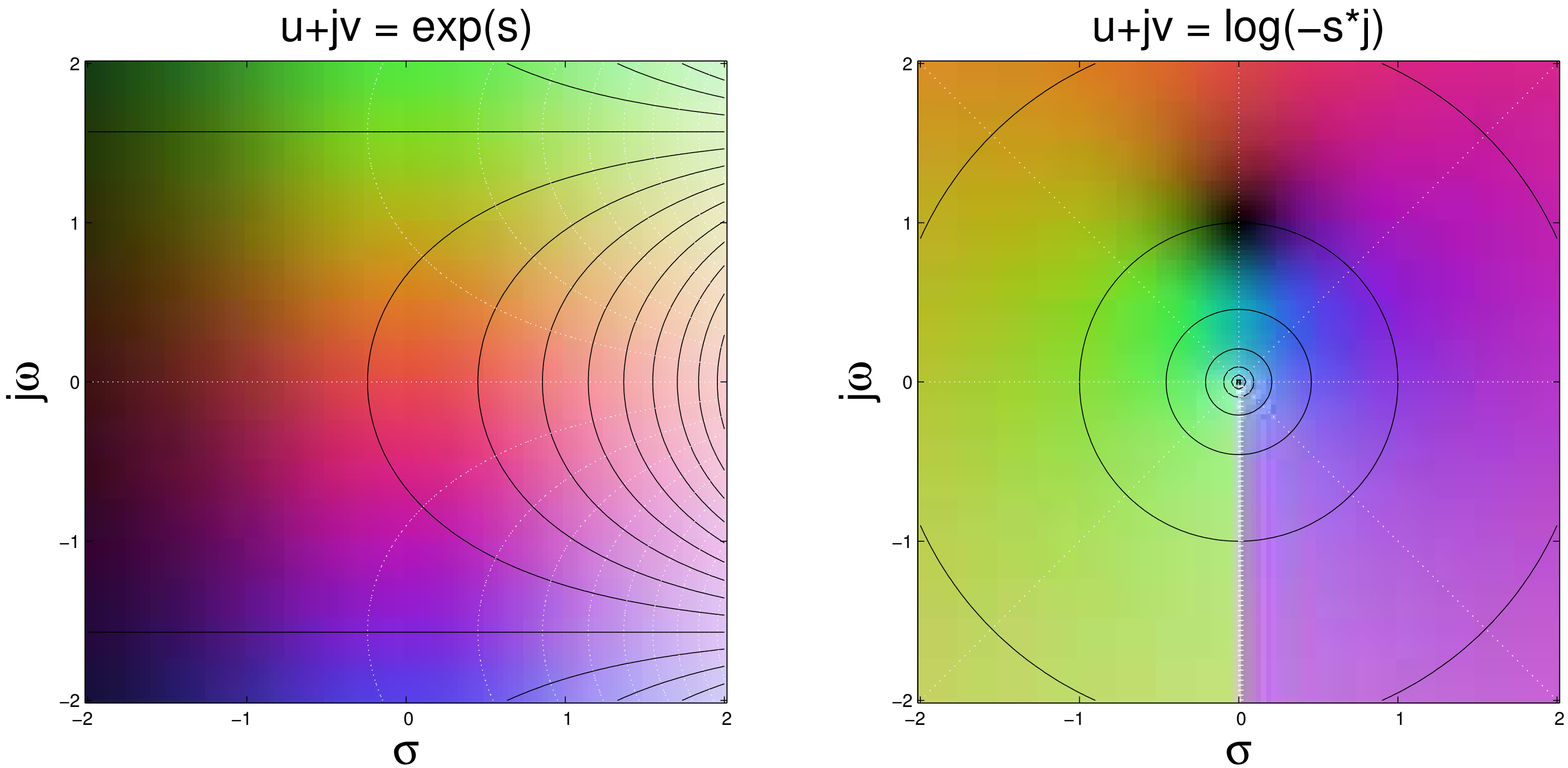
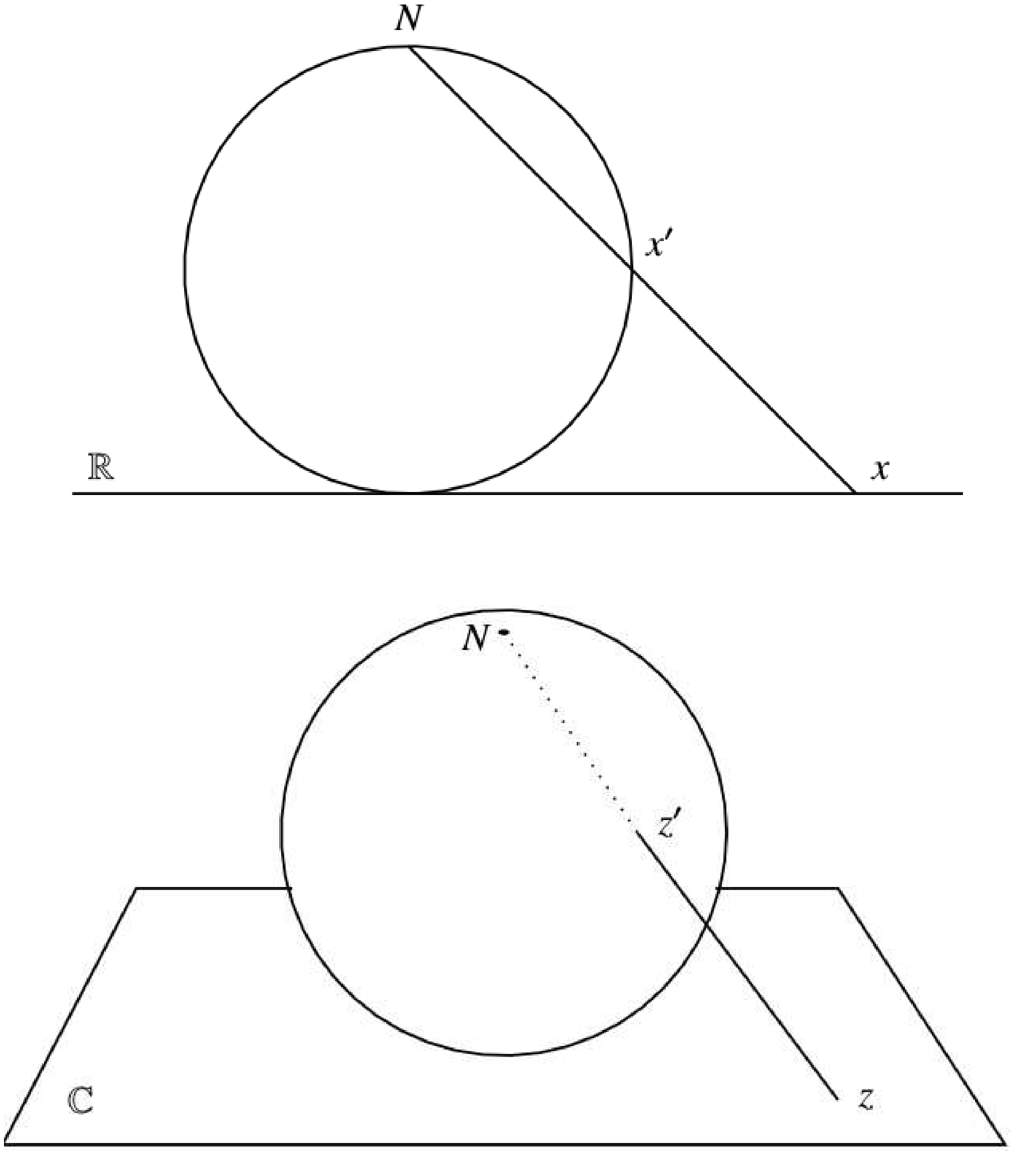


Figure: *Plots in the complex $z = x + y\jmath$:* *Left:* $e^{-s\jmath}$ *Right:* $\log(-s\jmath)$*, the inverse of the periodic* $e^{-s\jmath} = \cosh(-s\jmath) + \sinh(-s\jmath)$*, thus it has a branch cut, and a zero at* $s = \pi\jmath$ *(i.e.,* $\log(\pi\jmath = 0)$*).*

Riemann projection closes point $|z| \to \infty$ (i.e., $z' \to N$)



History of Acoustics, Music, Speech

- BC Pythagoras; Aristotle
- 16^{th} Mersenne, Marin 1588-1647; *Harmonie Universelle 1636*, *Father of acoustics*; Galilei, Galileo, 1564-1642; *Frequency Equivalence 1638*
- 17^{th} Newton, Hooke, Boyle
- 18^{th} Euler; d'Alembert; Gauss
- 19^{th} Fourier; Helmholtz; Kirchhoff; AG Bell; Lord Rayleigh

Stream 3b: WEEK 12-14, Lects 35-42

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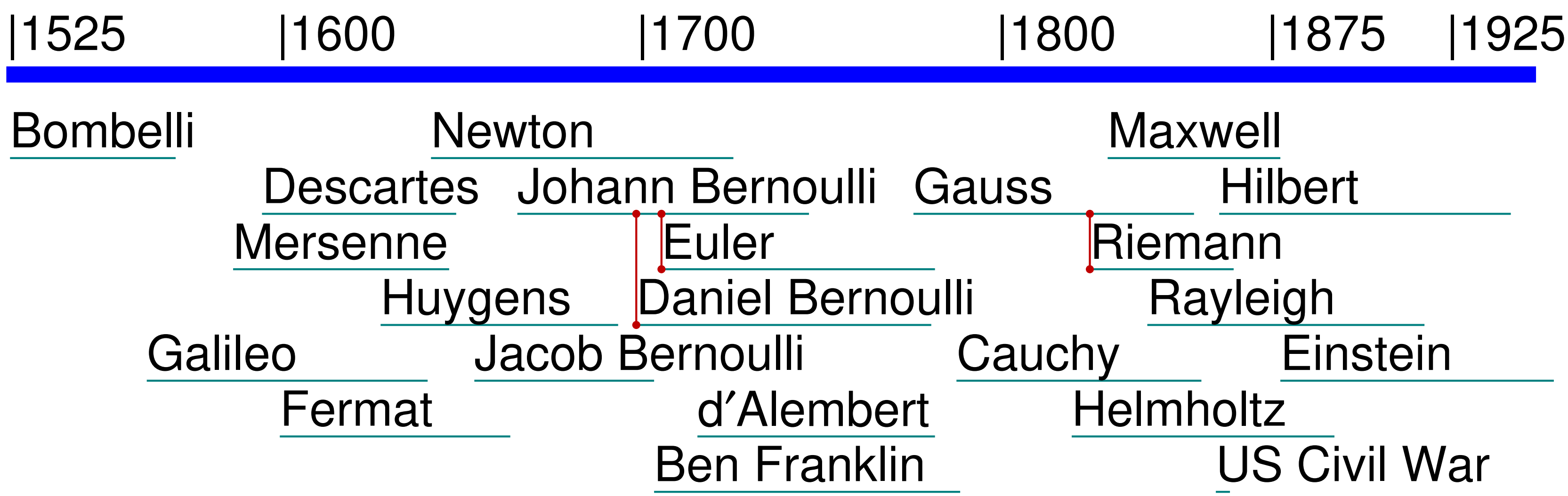
Time-line: Bombelli–Einstein, 16-20 centuries



- Bombelli discovers Diophantus' *Arithmetica* in Vatican library
  - $\Rightarrow$ Galileo, Descartes, Newton, Fermat, Bernoulli, Gauss, . . .
  - Johann teaches mathematics to Euler and Daniel
  - Euler technique dominates mathematics for 200 years (examples)
  - Gauss a close second: conceptual depth

Acoustics, Vector calculus and circuit theory

- Helmholtz Theorem: Vector field $F = -\nabla\Phi + \nabla \times A$
- Kirchhoff's Laws of circuit theory (similar to Newton's Laws)

von Helmholtz

Gustav Kirchhoff

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